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T R A C T S,

Physical and Mathematical.

CONTAINING,

An Explication of several important points
in PHYSICAL ASTRONOMY;

A N D,

A new method for ascertaining the Sun's
distance from the Earth, by the THEO-
RY OF GRAVITY.

By ·Dr MATTHEW STEWART,
Professor of Mathematics in the University of Edinburgh.

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M D C C L X I.

T R A C T

Physical and Mathematical

CONTAINING

A Treatise of Natural Philosophy
IN PHYSICAL ASTRONOMY;

AND

A new method for determining the
distances from the Earth to the fixed
stars of GREAT DISTANCE.

By DR. MATTHEW W. P. L. WILKINSON.



T O
THE RIGHT HONOURABLE,
JOHN EARL OF BUTE,
&c. &c. &c.

ONE OF HIS MAJESTY'S PRIN-
CIPAL SECRETARIES OF
STATE:

A JUDGE AND PATRON OF EVE-
RY ELEGANT AND USEFUL ART;

AN EARLY AND EMINENT PRO-
FICIENT IN THOSE MORE AB-
STRUSE SCIENCES WHICH THE
FOLLOWING TRACTS ARE IN-
TENDED TO ILLUSTRATE,

THEY ARE HUMBLY INSCRIBED BY
HIS LORDSHIP'S
MOST OBEDIENT AND
DEVOTED SERVANT,

MATTH. STEWART.

T O

THE RIGHT HONOURABLE

JOHN EARL OF BUTE

Sec. Sec. Sec.

ONE OF HIS MAJESTY'S
CHIEF SECRETARIES OF
STATE

A JUDGE AND PATRON OF
BY ELEGANT AND USEFUL ARTS

AN EARLY AND EMINENT PRO-
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STRACT SCIENCES WHICH THE
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THEY ARE NUMERALLY INSCRIBED BY

HIS LORDSHIP

MOST OBEYANT AND
DEVOTED SERVANT

MATTHEW STEWART

P R E F A C E.

IN the following tracts, several things in the doctrine of centripetal forces and physical astronomy are attempted to be explained, in a way somewhat different from the usual manner of considering these subjects. There is likewise, from the theory of gravity, a new method pointed out for determining the sun's distance from the earth. It depends on ascertaining the proportion of the centripetal force of the moon to the earth, to the mean solar force affecting the gravity of the moon to the earth. This, it is thought, may be done, both from the motion of the moon's apogee, and likewise from the motion of its nodes. To ascertain this force is a point which merits attention, as it affords the most exact way of finding the sun's distance from the earth.

It is well known, that the various methods hitherto attempted to solve this curious problem, have

vi P R E F A C E.

have failed, in a great measure, on account of the vast distance of the sun from the earth: but the method hinted at here, will give the solution the more accurately, the more distant the sun is from the earth; for it proceeds on the supposition that the distance of the sun from the earth is great.

There are several propositions, particularly in the second and third tracts, the use of which will not easily appear to the reader. They were intended to explain several things in the lunar theory which the author had in view, but which, as the book has swelled to a greater size than was at first expected, he was obliged to defer at present. If what is done here be found agreeable to the public, they may probably be the subject of another volume.

The last eight propositions of the fourth tract, (some of which were published in the second volume of the Edinburgh Essays, art. 6.), contain

P R E F A C E. vii

tain a very short, easy, and exact method, of finding the place of a planet in its orbit. This is a problem of the utmost consequence in astronomy, and several solutions of it have been given by mathematicians of the first rank. Their methods of solution require a considerable knowledge in the more difficult parts of mathematics; but the method here given, requires only the knowledge of the elementary parts, and of plain trigonometry.

As the reader is supposed to be well acquainted with Euclid's Elements, there are no references made to them; but as his knowledge of conic sections may not be so extensive, there are references made to the excellent treatise on this subject, by that great geometrician Dr Robert Simson Professor of Mathematics in the university of Glasgow, whose reputation is already great in the learned world, and must continue to increase, as long as true geometry is understood, or valued.

College of Edinburgh,
Aug. 20. 1761.

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College of Edinburgh
Aug 20. 1761.

TRACTS,
Physical and Mathematical.

TRACT I.
PROPOSITION I. Fig. 1.

LET ABC be any curve whose axis is AD: and let EFG be a curve line, so related to the curve ABC, that if from any point H, in AD, there be drawn HB perpendicular to AD, meeting the curve ABC in B, and the curve EFG in F; and if from B there be drawn Bf, a tangent to the curve ABC, in B, meeting AD in f; the rectangle fHF is to the square of HB, in a given ratio, suppose that of p to q : from A draw AE perpendicular to AD, meeting the curve EFG in E: the rectangle fHF will be double of the area AEFH.

Let AH be divided into indefinitely little
A parts,

parts, in the points K, L, M, &c.; draw KN, LO, MP, &c. parallel to HB, meeting the curve ABC in N, O, P, &c. and the curve EFG in Q, R, S, &c.; through B draw BT parallel to AD; and let the rectangle HBT be equal to the rectangle fHF , and join HT; through N, O, P, &c. draw NV, OX, PY, &c. parallel to BT or AD, meeting HT in V, X, Y, &c.; and let NZ, a tangent to the curve ABC in N, meet AD in Z; let NV meet BH in a , and OX meet BH, NK, in b , c .

Because the rectangle HBT is equal to the rectangle fHF , BT will be to HF as fH to HB; that is, (because the triangles fHB , NaB , may be considered as similar), as Na or KH to Ba ; therefore the rectangle aBT is equal to the rectangle KHF. Again, because the rectangle ZKQ is to the square of KN or Ha as p to q , that is, as the rectangle fHF to the square of HB, and the square of Ha is to the rectangle HaV as the square of HB to the rectangle HBT; therefore, by equality, the rectangle ZKQ will be to the rectangle HaV , as the rectangle fHF to the rectangle HBT; therefore the rectangle ZKQ will be equal to the rectangle HaV ; and

and therefore Va will be to KQ , as ZK to Ha or KN , that is, as oc to cN , that is, (because oc is equal to LK , and cN equal to ab), as LK to ab , therefore the rectangle baV is equal to the rectangle LKQ . Again, let PY meet BH in d , and the same way it is shown, that the rectangle dbX is equal to the rectangle MLR , and so on; therefore the sum of the rectangles aBT , baV , dbX , &c. is equal to the sum of the rectangles KHF , LKQ , MLR , &c.: but the sum of the rectangles aBT , baV , dbX , &c. may be considered as equal to the triangle HBT , and the sum of the rectangles KHF , LKQ , MLR , &c. may be considered as equal to the figure $AEFH$; therefore the triangle HBT is equal to the figure $AEFH$: but because the rectangle fHF is equal to the rectangle HBT , the rectangle fHF will be double of the triangle HBT ; and therefore the rectangle fHF will be double of the Area $AEFH$. Q. E. D.

P R O P. H. Fig. 2.

Let there be a curve ABC , whose axis is AD ;

A 2

and

4 TRACTS, PHYSICAL Tr. I.

and let EFG be a curve, whose asymptote is AH , perpendicular to AD ; let the curves ABC , EFG , be so related to each other, that if from any points B , C , in the curve ABC , there be drawn BK , CD , perpendicular to AD , meeting AD in K , D , and the curve EFG in F , G ; and BL , CM , tangents to the curve ABC in B , C , meet AD in L , M ; the rectangle MDG is to the rectangle LKF as CD to BK ; the figure $HADGE$ will be equal to the rectangle MDG .

From F draw FN parallel to BL , meeting AD in N ; and let CD be divided into indefinitely little parts in O , P , Q , R , S , &c.; draw OT , PV , QX , RY , SZ , &c. parallel to AD , meeting the curve ABC in T , V , X , Y , Z , &c.; and draw Ta , Vb , Xc , Yd , Ze , &c. parallel to CD , meeting AD in a , b , c , d , e , &c. and the curve EFG in f , g , h , i , &c.; and let Tm , Vn , tangents to the curve ABC , in T , V , meet AD in m , n ; let VP meet Ta in p .

Because FN is parallel to BL , BK will be to KL as FK to KN ; therefore the rectangle

angle

angle BKN will be equal to the rectangle LKF; and because the rectangle MDG is to the rectangle LKF as CD to BK, that is, as the rectangle contained by CD, KN, to the rectangle BKN; and the rectangle LKF is equal to the rectangle BKN; therefore the rectangle MDG will be equal to the rectangle contained by CD, KN. The same way it is shown, that the rectangle *maf* is equal to the rectangle contained by *aT*, KN; and so on. Again, because the rectangle MDG is equal to the rectangle contained by CD, KN; DG will be to KN as CD to DM, that is, as CO to OT or *Da*; therefore the rectangle *GDa* is equal to the rectangle contained by CO, KN: and because the rectangle *maf* is equal to the rectangle contained by *Ta*, KN; *af* will be to KN as *Ta* to *am*, that is, as *Tp* to *pV*, or *ab*; therefore the rectangle *fab* is equal to the rectangle contained by *Tp*, KN; that is, equal to the rectangle contained by OP, KN. The same way it is shown, that the rectangle *gbc* is equal to the rectangle contained by PQ, KN; and so on: therefore
the

the sum of the rectangles GDa , fab , gbc , is equal to the rectangle contained by KN , and the sum of CO , OP , PQ , &c.: but the sum of the rectangles GDa , fab , gbc , &c. may be considered as equal to the figure $HADGE$; and the sum of CO , OP , PQ , &c. is equal to the line CD ; therefore the figure $HADGE$ is equal to the rectangle contained by CD , KN , that is, equal to the rectangle MDG . Q. E. D.

P R O P. III. Fig. 3.

Let a body descend in the line AB , from the place A , and supposing any centripetal force whatever; let CDE be a curve, such, that if from any two points F , G , in AB , there be drawn FD , GE , perpendicular to AB , meeting the curve in D , E , the centripetal force at F is to the centripetal force at G as FD to GE ; from A draw AC , perpendicular to AB , meeting the curve in C ; the velocity at F will be to the velocity at G , in the subduplicate ratio of the area $ACDF$, to the area $ACEG$.

Let

Let AHK be a curve, such, that if from any points F, G , in AB , there be drawn FH, GK , perpendicular to AB , meeting the curve AHK in H, K ; FH is to GK as the velocity at F to the velocity at G ; and let HL, KM , tangents to the curve AHK , in H and K , meet AB in L, M ; let FN, GO , be spaces passed over, in an indefinitely little part of time, by the body when at F and G ; through N, O , draw NP, OQ , perpendicular to AB , meeting the curve AHK in P, Q ; draw HR, KS , parallel to AB , meeting NP, OQ , in R, S ; and let the rectangle KGT be equal to the rectangle contained by FH, GE .

Because FH is to NP as the velocity at F to the velocity at N , and GK to OQ as the velocity at G to the velocity at O ; PR will be to QS as the increment of the velocity at F to the increment of the velocity at G , that is, as the centripetal force at F to the centripetal force at G , that is, as FD to GE ; therefore FD is to GE as PR to QS ; and because the spaces FN, GO , are passed over in the same time, the velocity at F will be to the velocity at G as

FN

FN to GO, that is, as HR to KS, that is,
 FH is to GK as HR to KS. Because HR
 is to RP as LF to FH, that is, as the rect-
 angle LFD to the rectangle DFH; and
 PR is to QS, as FD to GE, that is, as the
 rectangle DFH to the rectangle contained
 by FH, GE, or the rectangle KGT; HR
 will be to QS as the rectangle LFD to the
 rectangle KGT: but QS is to SK as KG
 to GM, that is, as the rectangle KGT to
 the rectangle MGT; therefore HR is to
 KS as the rectangle LFD to the rectangle
 MGT; therefore FH is to GK as the rect-
 angle LFD to the rectangle MGT: but
 the square of FH is to the rectangle contain-
 ed by FH, GK, as FH to GK, that is, as
 the rectangle LFD to the rectangle MGT;
 and the rectangle contained by FH, GK, is
 to the square of GK, as FH to GK, that is,
 (because the rectangle KGT is equal to the
 rectangle contained by FH, GE), as GT to
 GE, that is, as the rectangle MGT to the
 rectangle MGE; therefore the square of
 FH is to the square of GK as the rectangle
 LFD to the rectangle MGE; therefore
 [prop. 1.] the rectangle LFD will be dou-
 ble

ble of the area ACDF, and likewise the rectangle MGE will be double of the area ACEG; therefore the rectangle LFD will be to the rectangle MGE as the area ACDF to the area ACEG; therefore the square of FH will be to the square of GK as the area ACDF to the area ACEG; therefore FH will be to GK in the subduplicate ratio of the area ACDF to the area ACEG: but FH is to GK as the velocity at F to the velocity at G; therefore the velocity at F is to the velocity at G in the subduplicate ratio of the area ACDF to the area ACEG.
Q. E. D.

PROP. IV. Fig. 4.

Let a body descend in the line AB, from the place A; and supposing any centripetal force whatever, let CDE be a curve, such, that if from any Points F, G, in AB, there be drawn FD, GE, perpendicular to AB, meeting the curve CDE in D, E, GE is to FD as the velocity at F to the velocity at G; let AH, perpendicular to AB, be the asymptote of the curve CDE; the time of
B *descent*

descent from A to F will be to the time of descent from A to G as the figure HAFDC to the figure HAGEC.

Let AKL be a curve, such, that drawing FK, GL, perpendicular to AB, meeting the curve AKL in K, L, FK will be to GL as the time of descent from A to F to the time of descent from A to G: let KM, LN, tangents to the curve AKL at K, L, meet AB in M, N; let FO, GP, be spaces passed over by the body in an indefinitely little part of time; when at F, G, draw OQ, PR, perpendicular to AB, meeting the curve AKL in Q, R; draw KS, LT, parallel to AB, meeting OQ, RP, in S, T.

Because PR is to GL as the time of descent from A to P to the time of descent from A to G, and OQ to FK as the time of descent from A to O to the time of descent from A to F, RT will be to QS as the time of descent from G to P to the time of descent from F to O: but the time of descent from G to P is equal to the time of descent from F to O; therefore RT, QS, are equal. Because the spaces GP, FO,

are

are passed over in equal times, the velocity at G will be to the velocity at F as GP to FO, that is, as LT to KS: but DF is to GE as the velocity at G to the velocity at F; therefore DF is to EG as LT to KS: but because the rectangle MFD is to the rectangle contained by MF, EG, as FD to EG, that is, as LT to KS, and the rectangle contained by MF, EG, is to the rectangle contained by FK, EG, as MF to FK, that is, as KS to SQ; therefore, by equality, the rectangle MFD will be to the rectangle contained by FK, EG, as LT to QS, that is, as LT to TR, that is, as NG to GL, that is, as the rectangle NGE to the rectangle LGE; and alternating, the rectangle MFD will be to the rectangle NGE as the rectangle contained by FK, EG, to the rectangle LGE, that is, as FK to GL: therefore [prop. 2.] the figure HAFDC is equal to the rectangle MFD, and the figure HAGEC equal to the rectangle NGE; therefore FK will be to GL as the figure HAFDC to the figure HAGEC: but FK is to GL as the time of descent from A to F to the time of descent

from A to G ; therefore the time of descent from A to F is to the time of descent from A to G as the figure HAFDC to the figure HAGEC. 2. E. D.

P R O P. V. Fig. 5.

Suppose a body to descend from the point A, in the line AB; and let ACDE be a curve, so related to the line AB, that if from any two points F, G, in AB, there be drawn FC, GD, perpendicular to AB, meeting the curve ACD in C, D, the velocity at F is to the velocity at G in the subduplicate ratio of FC to GD: let CH, DK, tangents to the curve ACD, in the points C, D, meet AB in H, K; the centripetal force at F will be to the centripetal force at G as the rectangle contained by FC, GK, to the rectangle contained by GD, HF.

Let LMN be a curve, so related to AB, that if from any two points F, G, in AB, there be drawn FM, GN, perpendicular to AB, meeting the curve LMN in M, N, the centripetal force at F is to the centripetal

tal

tal force at G as FM to GN : in FB take FO indefinitely little, and draw OP parallel to FC , meeting the curve ACD in P ; and in GB take GQ equal to FO ; and draw QR parallel to GD , meeting the curve ACD in R ; draw CS , DT , parallel to AB , meeting OP , QR , in S , T ; let OP , QR meet the curve LMN in V , X ; and draw AL perpendicular to AB , meeting the curve LMN in L .

Because the velocity at F is to the velocity at G in the subduplicate ratio of FC to GD , and the velocity at F is [3.] to the velocity at G in the subduplicate ratio of the space $LAFM$ to the space $LAGN$; therefore the space $LAFM$ will be to the space $LAGN$ as FC to GD ; and therefore the curves ACD , LMN , are so related to each other, that if from any two Points F , G , in AB , there be drawn FC , GD , perpendicular to AB , meeting the curve ACD in C , D , and likewise meeting the curve LMN in M , N ; the space $LAFM$ will be to the space $LAGN$ as FC to GD : therefore the space $LAOV$ is to the space $LAFM$ as OP to FC ; therefore, by division, the
rectangle

rectangle MFO will be to the space LAFM as SP to FC: but the space LAFM is to the space LAGN as FC to GD; therefore, by equality, the rectangle MFO is to the space LAGN as PS to GD: but because the space LAGN is to the space LAQX as GD to QR, the space LAGN will be to the rectangle NGQ as GD to RT; therefore, (because the rectangle MFO is to the space LAGN as PS to GD, and the space LAGN is to the rectangle NGQ as GD to RT), by equality, the rectangle MFO will be to the rectangle NGQ as PS to RT: but because FO, GQ, are equal, FM is to GN as the rectangle MFO to the rectangle NGQ; therefore FM is to GN as PS to RT. Because FO is indefinitely little, the triangles HFC, CSP, may be considered as similar; therefore HF will be to FC as CS or FO to PS; therefore the rectangle contained by PS, HF, will be equal to the rectangle CFO. The same way it is shown, that the rectangle contained by RT, GK, is equal to the rectangle DGQ; therefore the rectangle contained by PS, HF, will be to the rectangle contained by RT,

RT, GK, as the rectangle CFO to the rectangle DGQ, that is, as FC to GD, that is, as the rectangle contained by CF, GK, to the rectangle DGK; therefore the rectangle contained by PS, HF, will be to the rectangle contained by CF, GK, as the rectangle contained by RT, GK, to the rectangle DGK, that is, as RT to DG, that is, as the rectangle contained by RT, HF, to the rectangle contained by DG, HF; therefore the rectangle contained by PS, HF, will be to the rectangle contained by RT, HF, as the rectangle contained by CF, GK, to the rectangle contained by DG, HF; that is, PS will be to RT as the rectangle contained by CF, GK, to the rectangle contained by DG, HF. But FM is to GN as PS to RT; therefore FM is to GN as the rectangle contained by CF, GK, to the rectangle contained by DG, HF. But the centripetal force at F is to the centripetal force at G as FM to GN; therefore the centripetal force at F is to the centripetal force at G as the rectangle contained by CF, GK, to the rectangle contained by DG, HF. Q. E. D.

PROP.

PROP. VI. Fig. 6.

If a body acted upon by a centripetal force, tending to a fixt centre, describes a curve line, the angular velocity of the body round the fixt centre will be reciprocally as the square of the distance of the body from the centre.

Let ABC be any curve described by a body that is acted upon by a centripetal force, tending to the centre S; draw SA, SB, to any two points A, B, in the curve; the angular velocity of the line SA round the point S is to the angular velocity of the line SB round the point S as the square of SB to the square of SA.

Let AD, BE, be described in indefinitely little equal parts of time; join SD, SE; with the centre S, and distance SD, describe a circle meeting SA, SB, SE, in F, G, H; and with the centre S, and distance SE, describe a circle meeting SB in K.

Because AD, BE, are described in equal times, the triangles ASD, BSE, will be equal;

Plate 1.

Fig. 1.

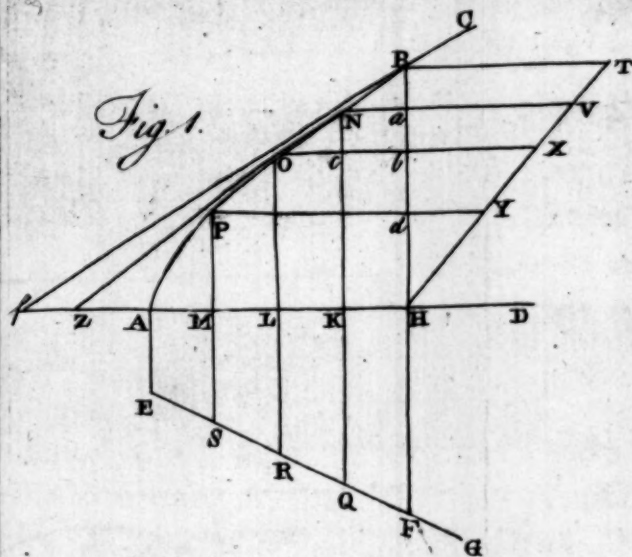


Fig. 2.

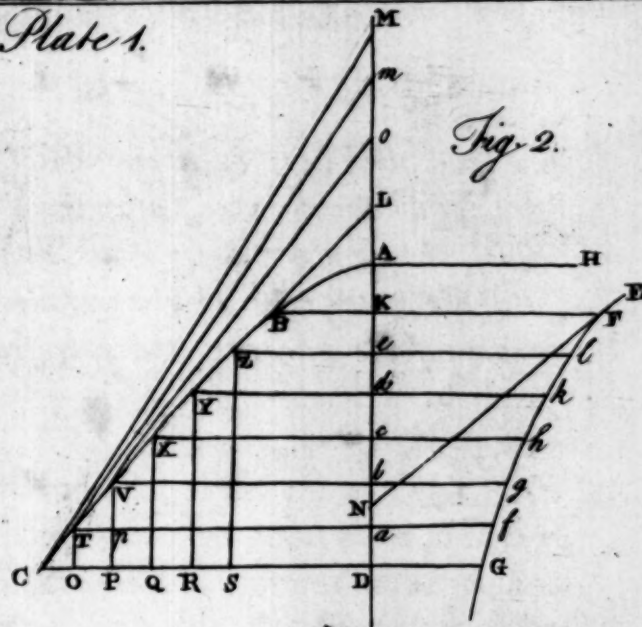


Fig. 3.

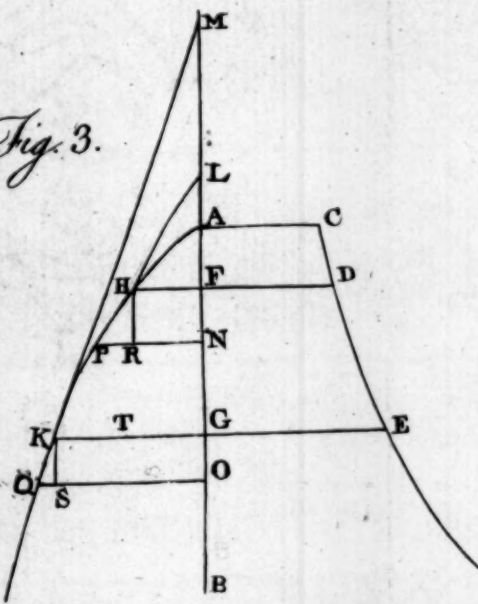
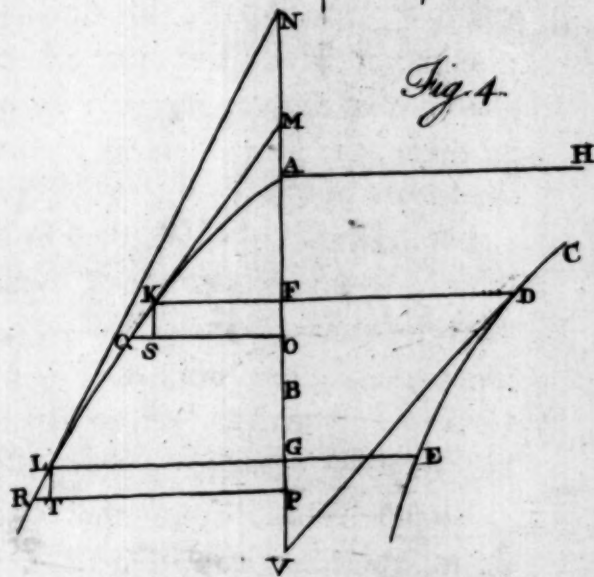


Fig. 4.



qual ; therefore DF will be to EK as BS to SA, that is, as the square of BS to the rectangle ASB : and EK is to GH as ES to SH, that is, as BS to SA, that is, as the rectangle ASB to the square of SA ; therefore, by equality, DF is to GH as the square of SB to the square of SA ; and therefore the angular velocity of the line SA round the point S is to the angular velocity of the line SB round the point S as the square of SB to the square of SA.
 Q. E. D.

P R O P. VII. Fig. 7.

Let SA, SB, be two right lines intersecting each other in the point S; and let AC, BD, be perpendicular to SA, SB; and let AE, BF, be indefinitely little parts of the lines AC, BD, described by bodies moving in the lines AC, BD, in an indefinitely little part of time; let SB be to SG as the square of BF to the square of AE: the centrifugal force at B will be to the centrifugal force at A as AS to SG; that is, the force of the body at B, to fly off from the centre S, will
 C be

be to the force of the body at A, to fly off from the centre S, as AS to SG.

Join SE, SF; with the centre S, and distance SA, describe a circle meeting SE in H, K; and with the centre S, and distance SB, describe a circle meeting SF in L, M. Because the square of BF is equal to the rectangle MFL, and the square of AE equal to the rectangle KEH; the square of BF will be to the square of AE as the rectangle MFL to the rectangle KEH. But because BF, AE, are indefinitely little, MF, ML, may be considered as equal, and likewise KE, KH, may be considered as equal; therefore the square of BF will be to the square of AE as the rectangle MLF to the rectangle KHE, that is, as the rectangle SLF to the rectangle SHE; but the square of BF is to the square of AE as BS or SL to SG, that is, as the rectangle SLF to the rectangle contained by SG, FL; therefore the rectangle SLF will be to the rectangle SHE as the rectangle SLF to the rectangle contained by SG, FL; therefore the rectangle SHE is equal to the rectangle contained

contained by SG, FL; therefore FL is to EH as AS to SG: but FL is to EH as the centrifugal Force at B to the centrifugal force at A; therefore the centrifugal force at B is to the centrifugal force at A as AS to SG. Q. E. D.

P R O P. VIII. Fig. 8.

Let ABC be any curve, described by a body that is acted upon by a centripetal force, tending to the centre S; the centrifugal force at any point B in the curve will be to the centrifugal force at any other point A in the curve as the cube of SA to the cube of SB.

Suppose the body, in an indefinitely little part of time, to describe the part AD; and let BE be a part described by the body, in a time equal to that in which the part AD was described; draw DK, EL, perpendicular to SA, SB, meeting SA, SB, in K, L; complete the parallelograms AKDF, BLEG; and let the square of BG be to the square of FA as BS to SH: join SD, SE, SF, SG.

Because AD, BE, are indefinitely little, they may be considered as straight lines; the force acting in the direction AD may be resolved into the forces AF, AK; likewise the force acting in the direction BE may be resolved into the forces BG, BL. Because AD, BE, are described in equal times, the triangles ADS, BES, will be equal; and therefore the triangles SAF, SBG, will be equal; therefore AS will be to SB as BG to AF; therefore the square of AS will be to the square of SB as the square of BG to the square of AF, that is, as BS to SH; therefore the parallelopiped whose base is the square of AS, and altitude SH, will be equal to the cube of SB; therefore the cube of AS will be to the cube of SB as the cube of AS to the parallelopiped whose base is the square of AS, and altitude SH, that is, as AS to SH: but the centrifugal force at B is to the centrifugal force at A [7.] as AS to SH; therefore the centrifugal force at B is to the centrifugal force at A as the cube of SA to the cube of SB. Q. E. D.

PROP.

PROP. IX. *Fig. 9.*

Supposing a body acted upon by a centripetal force tending to the centre S, to be projected at the point A, in the direction AB, perpendicular to AS; and suppose in an indefinitely little part of time the body would describe AC by the projectile force alone; let AD be the space that would be described in the same time by the centripetal force tending to the centre S: the centrifugal force at the point A, arising from the projectile force in the direction AB, will be to the centripetal force at the point A, tending to the centre S, as the square of AC to twice the rectangle SAD.

Join SC; with the centre S, and distance SA, describe a circle meeting SC in E, F. Because EC is to AD as the rectangle FCE to the rectangle contained by FC, AD, and the rectangle FCE is equal to the square of AC; therefore CE is to AD as the square of AC to the rectangle contained by FC, AD: but because CE is indefinitely little, CF,

CF, FE, may be considered as equal; therefore the rectangle contained by FC, AD, may be considered as equal to the rectangle contained by FE, AD, that is, equal to twice the rectangle SAD; therefore CE will be to AD as the square of AC to twice the rectangle SAD: but CE is to AD as the centrifugal force at A to the centripetal force at A; therefore the centrifugal force at A is to the centripetal force at A as the square of AC to twice the rectangle SAD. $\mathcal{Q}$. E. D.

P R O P. X. Fig. 10.

Supposing a body, in an indefinitely little part of time, to describe the part AD with an uniform velocity, and a body acted upon by a centripetal force tending to the centre S, to describe the part AE, in the same time: the velocity at E will be to the velocity in AD as twice AE to AD.

Because the time in which AE is described is indefinitely little, the centripetal force may be considered as acting uniformly; therefore the body, with the velocity acquired

acquired at E, acting uniformly, would describe the double of AE in the same time that AE was described, that is, in the same time that AD was described with an uniform velocity; therefore the velocity at E is to the velocity in AD as twice AE to AD. Q. E. D.

P R O P. XI. Fig. 11.

Supposing a body acted upon by a centripetal force tending to the centre S, to descend in the straight line AS from the point A; let there be a curve CDE, such, that if from any two points A, B, in AS, there be drawn AC, BD, perpendicular to AS, meeting the curve CDE, in C, D, the centripetal force at A is to the centripetal force at B as AC to BD: let the body at the point B be projected in the direction BF, perpendicular to SB, with the velocity that would be acquired in falling from A to B; the centrifugal force at B, arising from the projectile force in the direction BF, will be to the centripetal force at B, tending to the centre S, as twice the space CABD to the rectangle SBD.

Let

Let the body, in an indefinitely little part of time, descend from B to H by the centripetal force; and in an equal time, let the body, by the projectile force in the direction BF, describe BG: draw HK parallel to BD, meeting the curve CDE in K; and let the rectangle DBL be equal to the space CABD.

Because the velocity at B, by falling from A to B, is [prop. 3.] to the velocity at H, by falling from B to H, in the subduplicate ratio of the space CABD to the space DBHK; and the space CABD is equal to the rectangle DBL, and the rectangle DBH equal to the space DBHK, because BH is indefinitely little; the velocity at B, by falling from A to B, will be to the velocity at H, by falling from B to H, in the subduplicate ratio of the rectangle DBL to the rectangle DBH, that is, in the subduplicate ratio of LB to BH; and the velocity at B, by falling from A to B, is equal to the velocity in BF: therefore the velocity in BF is to the velocity at H, in the subduplicate ratio of LB to BH: but the velocity in BF is [prop. 10.] to the velocity at H

as

as BG to twice BH; therefore BG is to twice BH in the subduplicate ratio of LB to BH; therefore the square of BG is to four times the square of BH as LB to BH, that is, as four times the rectangle LBH to four times the square of BH; therefore the square of BG is equal to four times the rectangle LBH: and because the centrifugal force at B, arising from the projectile force in the direction BF, is to the centripetal force at B [9.] as the square of BG to twice the rectangle SBH, the centrifugal force at B will be to the centripetal force at B as four times the rectangle LBH to twice the rectangle SBH; that is, as twice LB to BS; that is, as twice the rectangle LBD to the rectangle SBD; that is, as twice the area CABD to the rectangle SBD. Q. E. D.

COR. In SB take any point *a*, and draw *ac* parallel to BD, meeting the curve CDE in *c*; let the body at the point B be projected in the direction BF, perpendicular to SB, with the velocity that would be acquired in falling from B to *a*: the centri-

D

fugal

fugal force at B, arising from the projectile force in the direction BF, will be to the centripetal force at B as twice the area caBD to the rectangle SBD.

This may be demonstrated in the same way as the proposition.

P R O P. XII. *Fig. 12.*

Let a body acted upon by a centripetal force tending to the centre S, describe the curve ABC; and let AD, BE, be tangents to the curve at any points A, B, in the curve; let SD, SE, perpendicular to AD, BE, meet AD, BE, in D, E: the velocity of the body at A will be to the velocity at B as ES to SD.

Join SA, SB; and let AF, BG, be parts of the curve described in indefinitely little equal parts of time; join SF, SG: Because AF, BG, are described in equal times, the triangles ASF, BSG, will be equal; therefore AF will be to BG as SE to SD: but AF is to BG as the velocity at A to the velocity at B; therefore the velocity
at

at A is to the velocity at B as ES to SD.

Q. E. D.

P R O P. XIII. Fig. 13.

If a body acted upon by any centripetal force is any how moved, and another body descends or ascends in a right line, and their velocities be equal in any one case of equal altitudes; their velocities will be also equal at all equal altitudes.

Let a body descend from A through D and E to the centre C; and let another body move from V in the curve VBk; from the centre C, with any distances indefinitely near to each other, describe the concentric circles DB, EK, meeting the right line AC in D, E, and the curve VBk in B, K; draw CB, meeting KE in N; and on BK let fall the perpendicular NT; and imagine the bodies at D and B to have equal velocities: Then, because the distances CD, CB, are equal, the centripetal forces in D and B will be also equal. Let these forces be expressed by the equal lines DE,
D 2 BN;

BN; and let the force BN be resolved into the forces NT and TB: the force NT, acting in the direction of the line NT, perpendicular to the path of the body, will not at all affect or change the velocity of the body in that path, but only draw it aside from a rectilinear course, and make it deflect perpetually from the tangent of the orbit, and proceed in the curvilinear path BTK; that whole force therefore will be employed in producing this effect; but the other force BT, acting in the direction of the course of the body, will be all employed in accelerating it: and because B and K are indefinitely near to each other, and likewise D and E indefinitely near to each other, the accelerating force from B to K, and from D to E, may be considered as acting uniformly; therefore the accelerations of the bodies in D and B produced in equal times are as the lines DE, BT; therefore the velocity at E will be to the velocity at T as DE or BN to BT: but because the angle BNK is right, BN is to BT in the subduplicate ratio of BK to BT; therefore the velocity at E is to the velocity at T, in
the

the subduplicate ratio of BK to BT : but the velocity at K is to the velocity at T, in the subduplicate ratio of BK to BT ; therefore the velocity at E is to the velocity at T as the velocity at K to the velocity at T ; therefore the velocity at K is equal to the velocity at E. Q. E. D.

P R O P. XIV. Fig. 14.

Let there be a semicircle whose diameter is AB, and centre C ; bisect CB in D, and draw DE perpendicular to AB, meeting the semicircle in E ; join AE : if from any point F, in the semicircle, there be drawn FG, perpendicular to AB, meeting AB in G, and AE in H, the square of DE will be greater than the rectangle FGH.

Let EK, a tangent to the semicircle at E, meet AB in K, and FG in L : because of the circle, CK, CB, CD, are proportionals ; therefore, because CB is double of CD, CK will be double of CB ; therefore KB is equal to BC, that is, equal to AC ; and because BD is equal to DC, therefore AD, DK,

DK, are equal. Again, because the square of AD is to the rectangle DAG as DA to AG, that is, as DE to GH, that is, as the square of DE to the rectangle contained by DE, GH; alternating, the square of AD will be to the square of DE as the rectangle DAG to the rectangle contained by DE, GH: and because the rectangle AGK is to the rectangle DAG as GK to AD or DK, that is, as LG to ED, that is, as the rectangle LGH to the rectangle contained by DE, GH; the rectangle AGK will be to the rectangle LGH as the rectangle DAG to the rectangle contained by DE, GH; therefore the square of AD is to the square of DE as the rectangle AGK to the rectangle LGH: but the square of AD is greater than the rectangle AGK; therefore the square of DE is greater than the rectangle LGH; and therefore, because GL is greater than GF, the square of DE is greater than the rectangle FGH.

Q. E. D.

PRO P.

DK

PROP. XV. *Fig. 15.*

In the straight line AB, take BC so that AC may be triple of CB; and let D be any point in AB: the ratio of the cube of DA to the cube of AC will be less than the ratio of BC to BD.

Upon AB describe a semicircle AFB; and let CF, perpendicular to AB, meet the semicircle in F; join AF, and draw DG parallel to CF, meeting the semicircle in G, and AF in H; and let the rectangle KDH be equal to the square of CF.

Because the rectangle KDH is equal to the square of CF, HD will be to CF as CF to DK: but HD is to CF as AD to AC; therefore AD is to AC as CF to DK; therefore the square of AD is to the square of AC as the square of CF to the square of DK: but the cube of AD is to the parallelopiped whose base is the square of AC, and altitude AD, as the square of AD to the square of AC; therefore the cube of AD is to the parallelopiped whose base is the square

square of AC, and altitude AD, as the square of CF to the square of DK: but the parallelopiped whose base is the square of AC, and altitude AD, is to the cube of AC as AD to AC, that is, as the rectangle contained by AD, CB, to the rectangle ACB, or the square of CF; therefore the cube of AD is to the cube of AC as the rectangle contained by AD, CB, to the square of DK: but because [14.] the square of CF is greater than the rectangle GDH, the rectangle KDH will be greater than the rectangle GDH; therefore KD is greater than DG; therefore the square of DK is greater than the square of DG, or the rectangle ADB; therefore the ratio of the rectangle contained by AD, CB, to the square of DK, is less than the ratio of the rectangle contained by AD, CB, to the rectangle ADB, that is, less than the ratio of CB to BD: but the cube of AD is to the cube of AC as the rectangle contained by AD, CB, to the square of DK; therefore the ratio of the cube of AD to the cube of AC is less than the ratio of CB to BD.

Q. E. D.

P R O P.

Plate 2.

Fig. 5.

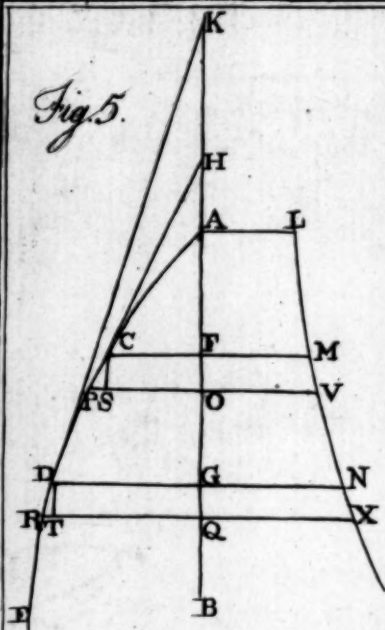


Fig. 6.

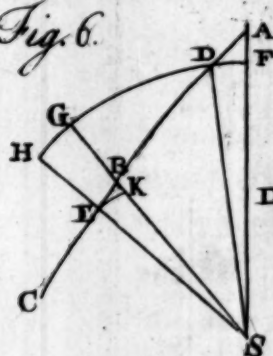


Fig. 7.

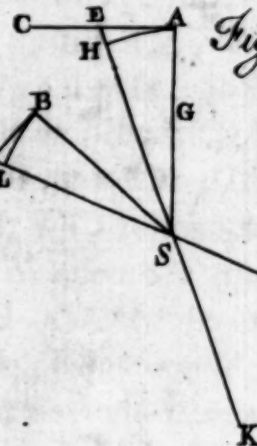


Fig. 8.

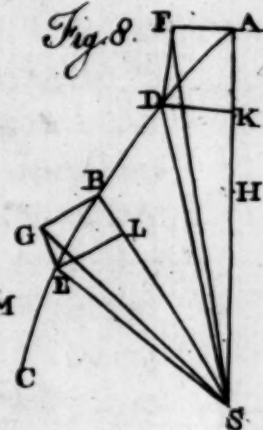


Fig. 9.

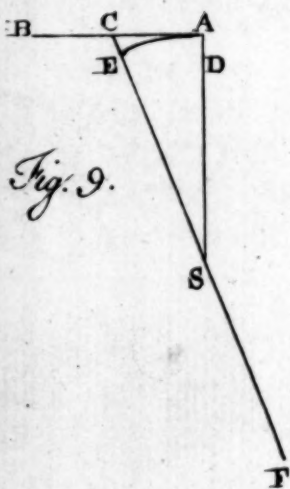


Fig. 10.

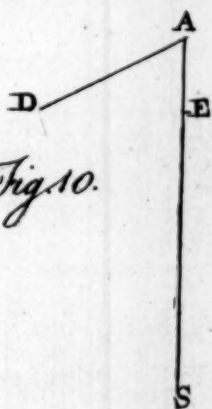


Fig. 11.

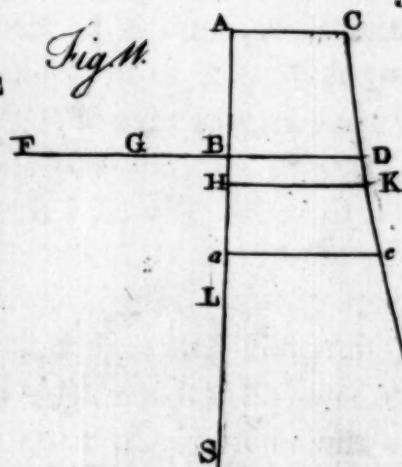
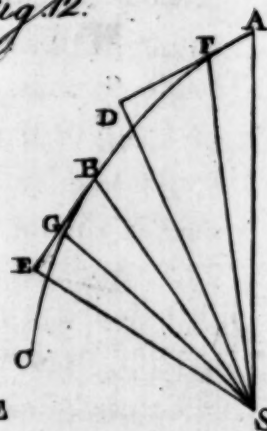


Fig. 12.



PROP. XVI. Fig. 16.

Let A be a given point in the straight line AB, and CDE a curve, such, that if from any two points C, D, in the curve CDE, there be drawn CF, DG, perpendicular to AB, meeting AB in F, G; the cube of AG is to the cube of AF as CF to DG: in AF take the point H so that AH may be quadruple of HF; join CH; CH will be a tangent to the curve CDE in the point C.

From any point K, in CH, draw KG perpendicular to AB, meeting AB in G, and the curve CDE in D.

Because AH is quadruple of HF, AF will be triple of HF; therefore [15.] the ratio of the cube of AG to the cube of AF will be less than the ratio of FH to HG; but because the cube of AG is to the cube of AF as CF to DG, and FH is to HG as CF to GK; therefore the ratio of CF to DG will be less than the ratio of CF to GK; therefore DG is greater than GK;

E

therefore

therefore the point K is without the curve CDE. The same way it is shown, that any other point in CH is without the curve CDE; therefore CH is a tangent to the curve CDE at the point C. Q. E. D.

P R O P. XVII. Fig. 17.

Let A be a given point in the straight line AB, and CDE a curve, such, that if from any two points D, E, in the curve CDE, there be drawn DF, EG, perpendicular to AB, meeting AB in F, G, DF will be to EG as the cube of AG to the cube of AF; draw AH perpendicular to AB: If from any two points D, E, in the curve, there be drawn DF, EG, perpendicular to AB, meeting AB in F, G; likewise there be drawn DK, EL, perpendicular to AH, meeting AH in K, L: the space DKLE will be triple of the space DFGE; likewise the excess of the rectangle AFD above the rectangle AGE will be double of the space DFGE.

Suppose KL to be divided into indefinitely

nitely little parts in M, N, &c.; draw MO, NP, &c. parallel to DK or EL, meeting the curve CDE in O, P, &c.; draw OV, PX, &c. parallel to DF or EG, meeting AB in V, X, &c.; let DF meet OM in q , OV meet NP in r , and PX meet EL in s ; and take in AB a point T, such, that AT may be quadruple of FT; and join DT.

Because AT is quadruple of FT, DT will [16.] be a tangent to the curve CDE at the point D; and because MO is indefinitely near to DK, the triangles DFT, DqO, may be considered as similar; therefore DF is to FT as Dq to qO; therefore the rectangle contained by Dq, FT, is equal to the rectangle contained by Oq, DF; that is, equal to the rectangle FO: but because AT is quadruple of TF, AF will be triple of FT; therefore the rectangle contained by AF, Dq, will be triple of the rectangle contained by FT, Dq; and therefore the rectangle DM will be triple of the rectangle FO. The same way it is shown, that the rectangle ON is triple of the rectangle VP, and the rectangle PL triple of the rectangle XE, and so on. But

the sum of the rectangles DM, ON, PL, &c. may be considered as equal to the figure DKLE, and the sum of the rectangles FO, VP, XE, &c. may be considered as equal to the figure DFGE; therefore the figure DKLE is triple of the figure DFGE.

Again, let DF meet EL in Y. Because the figure DKLE is triple of the figure DFGE, the rectangle DL will be equal to the rectangle FE, together with the double of the figure DFGE: let the rectangle FL be added to both, and the rectangle AD will be equal to the rectangle AE, together with the double of the figure DFGE; and therefore the excess of the rectangle AD above the rectangle AE, will be double of the figure DFGE: therefore the excess of the rectangle AFD, above the rectangle AGE is double of the figure DFGE.
Q. E. D.

P R O P. XVIII. Fig. 18.

Let S be a given point in the straight line AS, and CDE a curve, such, that if from any two points A, B, in AS, there be drawn
AC,

AC, BD, perpendicular to AS, meeting the curve CDE in C, D; BD is to AC as the cube of AS to the cube of SB; produce AS to F, and let AS, SF, be equal: the rectangle SAC is to the figure CABD as twice the square of SB to the rectangle ABF; likewise the rectangle SBD will be to the area CABD as twice the square of AS to the rectangle ABF.

Let the parallelopiped whose base is the square of AS, and altitude SG, be equal to the cube of SB: because BD is to AC as the cube of AS to the cube of SB; that is, as the cube of AS to the parallelopiped whose base is the square of AS, and altitude SG; that is, as AS to SG; the rectangle SAC will be equal to the rectangle contained by SG, BD; therefore the rectangle SBD will be to the rectangle SAC as the rectangle SBD to the rectangle contained by SG, BD; that is, as SB to SG: but because the parallelopiped whose base is the square of AS, and altitude SG, is equal to the cube of SB, the square of AS will be to the square of SB as SB to SG; therefore the

the rectangle SBD will be to the rectangle SAC as the square of AS to the square of SB; therefore the excess of the rectangle SBD above the rectangle SAC, will be to the rectangle SAC as the rectangle ABF to the square of SB: but [17.] the excess of the rectangle SBD above the rectangle SAC, is double of the figure CABD; therefore twice the figure CABD is to the rectangle SAC as the rectangle ABF to the square of SB; therefore twice the figure CABD is to twice the rectangle SAC as the rectangle ABF to twice the square of SB; and therefore the rectangle SAC is to the figure CABD as twice the square of SB to the rectangle ABF.

Again, because the rectangle SBD is to the rectangle SAC as the square of AS to the square of SB; that is, as twice the square of AS to twice the square of SB; and the rectangle SAC is to the area CABD as twice the square of SB to the rectangle ABF; therefore, by equality, the rectangle SBD will be to the area CABD as twice the square of AS to the rectangle ABF. *Q. E. D.*
 PROP.

P R O P. XIX. Fig. 19.

The law of centripetal force being given, and granting the quadrature of curves, it is required to find the trajectory in which a body will move, that sets out from a given place, with a given velocity, in the direction of a given right line.

Suppose AEFG to be the trajectory; with the centre C, and distance CA, describe the circle APQ; draw AY, CZ, perpendicular to CA, and let AY be equal to CA; with the asymptotes CA, CZ, let there be an hyperbola described passing through the point Y; let SRT be a curve, such, that drawing any line CFP, meeting the trajectory AEFG in F, and the circle APQ in P; and with the centre C, and distance CF, a circle be described meeting CA in K; and KR be drawn perpendicular to CA, meeting the curve SRT in R; the space SAKR will be double of the sector ACP. Let fall CB perpendicular to AB, the given right line in which the body

dy is projected; let AE, FG, be indefinitely little parts of the trajectory described in equal times; with the centre C, and distances CE, CF, CG, describe the circles EH, FK, GL, meeting AC in H, K, L; and let the circle GL meet CF in M; let DA be the space through which the body would fall, so as to acquire, at the point A, a velocity equal to the velocity with which the body is projected in the direction AB; let CX be to CK as the velocity that would be acquired in falling from D to K to the velocity that would be acquired in falling from D to A; draw La parallel to KR, meeting the curve SRT in *a*; join CG, meeting the circle APQ in Q; and let KR meet the hyperbola in V.

Because the body sets out at A to describe the trajectory with the same velocity that would be acquired in falling from D to A, and CF, CK, are equal, the velocity at F will be equal [13.] to the velocity that would be acquired in falling from D to K; therefore CX will be to CK as the velocity at F to the velocity at A; that is, as FG to AE:
but

but because AE, FG, are described in equal times, the triangles ACE, FCG, will be equal; therefore CK will be to CB as AE to GM; therefore, by equality, CX will be to CB as FG to GM; therefore the square of CX will be to the square of CB as the square of FG to the square of GM; and therefore, by division, the excess of the square of CX, above the square of CB, will be to the square of CB as the square of FM to the square of GM.

Again, because from the nature of the curve SRT, the space SALa is double of the sector ACQ, and the space SAKR double of the sector ACP, the space RKL^a will be double of the sector PCQ; therefore, because KL is indefinitely little, KL will be to PQ as CP or CA to KR; but PQ is to GM as CP or CA to CK; that is, because of the hyperbola, as KV to CA; therefore, by perturbate equality, KL is to GM as KV to KR; that is, FM is to GM as KV to KR; therefore the square of FM is to the square of GM as the square of KV to the square of KR: but the square of FM is to the square of GM as

F

the

the excess of the square of CX, above the square of CB, to the square of CB; therefore the square of KV will be to the square of KR as the excess of the square of CX, above the square of CB, to the square of CB. From whence may be deduced the following

CONSTRUCTION.

Draw AY, CZ, perpendicular to CA, and let AY be equal to CA; with the asymptotes CA, CZ, let there be an hyperbola described passing through the point Y; and let the body set out from A in the direction of the right line AB, with the velocity that would be acquired in falling from D to A; from C let fall CB perpendicular to AB; and let K be any point in AC; let CX be to CK as the velocity that would be acquired in falling from D to K to the velocity that would be acquired in falling from D to A; let SRT be a curve, such, that drawing KV perpendicular to AC, meeting the hyperbola in V, and the curve SRT in R, the square of KV will be to the square of KR as the excess of the square of CX, above

bove the square of CB, to the square of CB; let AY meet the curve SRT in S; with the centre C, and distance CA, describe the circle APQ; draw CP, such, that the area SAKR may be double of the sector CAP; with the centre C, and distance CK, describe a circle meeting CP in F: F will be a point in the trajectory.
Q. E. I.

P R O P. XX. *Fig. 20.*

Suppose a body acted upon by a centripetal force tending to the centre S to descend in the straight line AS; let there be a curve CDE, such, that if from any two points A, B, in AS, there be drawn AC, BE, perpendicular to AS, meeting the curve CDE in C, E; the centripetal force at A is to the centripetal force at B as AC to BE. Suppose the body at the point B to be projected in the direction BF, perpendicular to SB, with the velocity that would be acquired in falling from A to B; and let the centrifugal force at the point B, arising from the projectile force, be greater than the centri-

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petal

petal force at the point B; it is evident that the body will recede from the centre: in SA take SG, equal to the greatest distance the body will recede from the centre: the sum of the centrifugal forces that have acted upon the body while it receded from the centre, through the space BG, will be equal to the sum of the centripetal forces that have acted upon the body in the same time.

Let the rectangle SBF be double of the space CABE; and in BE take BH, so that the square of BH may be equal to the space CABE; draw SK perpendicular to SB; with the asymptotes SB, SK, let there be an hyperbola described passing through the point H; let BLM be the trajectory described by the body; suppose the body at any point L in the trajectory, and let LM be a part of the trajectory described in an indefinitely little part of time, and let BO be a part of the trajectory described in a time equal to the time the part LM was described; join SL, SM, SO; with the centre S, and distance SL, let there be a circle described meeting SA in G; and with

with the centre S, and distance SM, describe a circle meeting SL in N; draw GD perpendicular to SA, meeting the curve CDE in D, and the hyperbola in P.

Because LM, BO, are described in equal times, LM will be to BO as the velocity at L to the velocity at B; that is, as the velocity at G to the velocity at B: but the velocity at G is to the velocity at B [3.] in the subduplicate ratio of the space CAGD to the space CABE; therefore LM is to BO in the subduplicate ratio of the space CAGD to the space CABE; therefore the square of LM will be to the square of BO as the space CAGD to the space CABE; that is, as the space CAGD to the square of BH: but because the triangles SLM, SBO, are equal, (because LM, BO, are described in equal times), the square of BO will be to the square of MN as the square of SL, or SG to the square of SB; that is, (because of the hyperbola), as the square of BH to the square of GP; therefore, by equality, the square of LM is to the square of MN as the space CAGD to the square of GP: therefore it is evident, that if the
space

space CAGD be equal to the square of GP, the square of LM will be equal to the square of MN; and therefore LM, MN, must coincide; therefore SL or SG will be the greatest distance the body will recede from the centre S.

Let there be a curve FQ so related to the line SA, that if from any two points B, G, in SA, there be drawn BF, GQ, perpendicular to SA, meeting the curve FQ in F, Q; the cube of SG is to the cube of SB as BF to GQ; produce GS to R, and let SR be equal to SG.

Because [11.] the centrifugal force at B, arising from the projectile force in the direction BF, is to the centripetal force at B as twice the space CAGE to the rectangle SBE; and the rectangle SBF is double of the space CAGE; therefore the centrifugal force at the point B, arising from the projectile force in the direction BF, will be to the centripetal force at B as the rectangle SBF to the rectangle SBE; that is, as BF to BE; and because BE represents the centripetal force at B, therefore BF will represent the centrifugal force at B; and because
the

the centrifugal forces are, at different distances from the centre, to each other [8.] reciprocally as the cubes of the distances from the centre, therefore it is evident, that the sum of the centrifugal forces that have acted upon the body while it receded from the centre through the space BG will be to the sum of the centripetal forces that have acted upon the body in the same time as the space FBGQ to the space EBGD: because the square of BH is equal to the space CABE, and the square of GP equal to the space CAGD; therefore the space CABE will be to the space CAGD as the square of BH to the square of GP; that is, (because of the hyperbola), as the square of SG to the square of SB; therefore, conversely, the space CABE will be to the space EBGD as the square of SG to the rectangle GBR; therefore twice the space CABE, that is, the rectangle SBF, will be to the space EBGD as twice the square of SG to the rectangle GBR: but [18.] the rectangle SBF is to the space FBGQ as twice the square of SG to the rectangle GBR; therefore the rectangle SBF is to the space FBGQ

FBGQ as the rectangle SBF to the space EBGD; therefore the space FBGQ is equal to the space EBGD; therefore the sum of the centrifugal forces that have acted upon the body while it receded from the centre, through the space BG, will be equal to the sum of the centripetal forces that have acted upon the body in the same time.

COR. From this it is evident, that the sum of the centrifugal forces that act upon a body while it revolves from apsis to apsis, is equal to the sum of the centripetal forces that act upon it in the same time.

LEMMA. Fig. 21.

Let AB be the transverse axis of an ellipse or hyperbola; D, E, the foci; let KG be a tangent to the ellipse or hyperbola at the point G; from the foci D, E, draw DK, ER, perpendicular to the tangent, meeting the tangent in K, R: the rectangle contained by DK, ER, will be equal to the rectangle ADB.

Let

Let C be the centre; draw CS , ET , parallel to the tangent, meeting DK in S , T ; and join CK .

Because DE is bisected in C , DT will be bisected in S ; therefore the rectangle DKT , together with the square of TS or SD , will be equal to the square of KS ; therefore the rectangle DKT is equal to the excess of the square of KS above the square of SD ; that is, equal to the excess of the square of CK above the square of CD : but because [Con. 5. 25.] KC is equal to CB , the rectangle ADB will be the excess of the square of CK or CB above the square of CD ; therefore the rectangle DKT is equal to the rectangle ADB : because KT , ER , are equal, the rectangle DKT will be equal to the rectangle contained by DK , ER ; therefore the rectangle contained by DK , ER , is equal to the rectangle ADB . Q. E. D.

P R O P. XXI. Fig. 21.

Let there be an ellipse whose transverse axis is AB , and foci D , E ; produce DA to F ;
 G
 and

and let DF be equal to AB ; upon DF let there be a semicircle described; draw DG , DH , to any two points G , H , in the ellipse; and draw DK , DL , perpendicular to GK , HL , tangents to the ellipse at the points G , H ; with the centre D , and distances DG , DH , describe the circles GM , HN , meeting AB in M , N ; draw MO , NP , perpendicular to AD , meeting the semicircle in O , P ; and join DO , meeting NP in Q : DK will be to DL as PN to NQ .

Join EG ; and draw ER perpendicular to GK , meeting GK in R . Because the sum of DG , GE , is equal to AB ; that is, equal to DF ; and DG , DM , are equal; therefore GE , FM , are equal; and therefore DM is to MF as DG to GE : but because of the semicircle, the square of DM is to the square of MO as DM to MF ; therefore the square of DM will be to the square of MO as DG to GE : but because KR is a tangent to the ellipse at the point G , the angles DGK , EGR , will be equal; therefore, because the angles at K and R

are

are right, the triangles DGK, EGR, are equiangular; therefore DK will be to ER as DG to GE: but the square of DK is to the rectangle contained by DK, ER, as DK to ER; therefore the square of DK will be to the rectangle contained by DK, ER, as DG to GE; that is, as the square of DM to the square of MO; but [lem.] the rectangle contained by DK, ER, is equal to the rectangle ADB; therefore the square of DK will be to the rectangle ADB as the square of DM to the square of MO. The same way it is shown, that the square of DL is to the rectangle ADB as the square of DN to the square of NP; therefore, inversely, the rectangle ADB will be to the square of DL as the square of NP to the square of DN: but because the square of DK is to the rectangle ADB as the square of DM to the square of MO; that is, as the square of DN to the square of NQ; and the rectangle ADB is to the square of DL as the square of NP to the square of DN; therefore, by perturbate equality, the square of DK will be to the square of DL as the square of PN to the square of NQ;

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and therefore DK is to DL as PN to NQ.
Q. E. D.

COR. The square of DK will be to the rectangle ADB as DG to GE.

P R O P. XXII. Fig. 22.

Let there be a semicircle whose diameter is AB; and let ADE be a curve, such, that if from any two points D, E, in the curve ADE, there be drawn DC, EF, perpendicular to AB, meeting AB in C, F, and the semicircle in G, H, and joining BG, meeting FH in K; EF is to DC as the square of FH to the square of FK; the curve ADE will be an hyperbola: and let DL, EM, tangents to the hyperbola at D, E, meet AB in L, M, the rectangle contained by EF, CL, will be to the rectangle contained by DC, FM, as the square of BC to the square of BF.

Let AB be bisected in C; produce EF to O; and let FO, DC, be equal; through O draw PN parallel to AB, meeting AN; BP

BP drawn perpendicular to AB, in N, P;
and DC in R.

Because EF is to CD as the square of HF to the square of FK; and the square of HF (because of the semicircle) is equal to the rectangle AFB; and the square of FK equal to the square of FB, (because the square of GC is equal to the square of CB); therefore EF is to DC as the rectangle AFB to the square of FB; that is, as AF to FB; therefore, because DC, FO, are equal, EF will be to FO as AF to FB; and therefore, by composition, EO will be to OF as AB to BF; that is, as NP to PO; and because FO, AN, are equal, therefore EO will be to AN as NP to PO; therefore the rectangle EOP is equal to the rectangle ANP; and therefore the curve ADE is an hyperbola, whose asymptotes are PB, PN.

Again, let the tangents DL, EM, meet PN in N, Q; and let DC meet PN in R. Because the tangents DN, EQ, meet the asymptotes PN in N, Q; and DR, EO, drawn parallel to the asymptote PB, meet the asymptote in R, O; therefore NR, RP, will be equal; and likewise
QQ,

QQ , OP , will be equal: because EF is to FM as EO to OQ , that is, as EO to OP ; and EF is to FM as the rectangle contained by EF , LC , to the rectangle contained by FM , LC , and likewise EO is to OP as the square of EO to the rectangle EOP ; the rectangle contained by EF , LC , will be to the rectangle contained by FM , LC , as the square of EO to the rectangle EOP . Again, because LC is to CD as NR to RD , that is, as PR to RD , and LC is to CD as the rectangle contained by FM , LC , to the rectangle contained by DC , FM ; and PR is to RD as the rectangle PRD to the square of RD ; the rectangle contained by FM , LC , will be to the rectangle contained by DC , FM , as the rectangle PRD to the square of RD : but (because of the hyperbola) the rectangle PRD is equal to the rectangle EOP ; therefore the rectangle contained by FM , LC , will be to the rectangle contained by DC , FM , as the rectangle EOP to the square of RD : but because the rectangle contained by EF , LC , is to the rectangle contained by FM , LC , as the square of EO to the rectangle

rectangle

rectangle EOP; and the rectangle contained by FM, LC, is to the rectangle contained by DC, FM, as the rectangle EOP to the square of RD; therefore the rectangle contained by EF, LC, is to the rectangle contained by DC, FM, as the square of EO to the square of RD: but because the rectangle EOP is equal to the rectangle DRP, EO will be to DR as RP to PO, that is, as CB to BF; therefore the square of EO will be to the square of DR as the square of CB to the square of BF; and therefore the rectangle contained by EF, LC, is to the rectangle contained by DC, FM, as the square of BC to the square of BF. *Q. E. D.*

P R O P. XXIII. *Fig. 22.*

Suppose a body to descend or ascend in the line AB; let C, F, be any two points in AB; draw CG, FH, perpendicular to AB, meeting the semicircle described upon AB in G, H; join BG meeting FH in K: if the velocity at F be to the velocity at C as HF to FK, the centripetal force at F
will

will be to the centripetal force at C as the square of BC to the square of BF.

Let ADE be a curve, such, that if from any two points D, E, in the curve, there be drawn DC, EF, perpendicular to AB, meeting AB in C, F, and the semicircle described upon AB in G, H, and joining BG, meeting FH in K; EF is to CD as the square of HF to the square of FK; and let DL, EM, tangents to the curve at D, E, meet AB in L, M.

Because the square of HF is to the square of FK as EF to CD, HF will be to FK in the subduplicate ratio of EF to CD; but the velocity at F is to the velocity at C as HF to FK; therefore the velocity at F will be to the velocity at C, in the subduplicate ratio of EF to CD; therefore [5.] the centripetal force at F will be to the centripetal force at C as the rectangle contained by EF, CL, to the rectangle contained by DC, FM: but the rectangle contained by EF, CL, is [22.] to the rectangle contained by DC, FM, as the square of BC to the square of BF; therefore the centripetal force

Fig. 13.



Fig. 14.

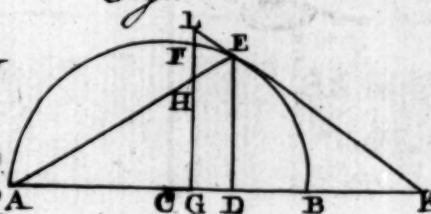


Fig. 15.

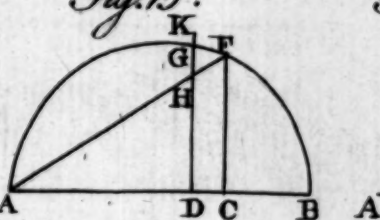


Fig. 16.

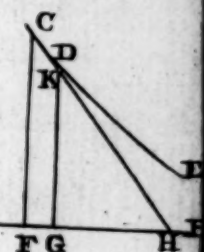


Fig. 18.

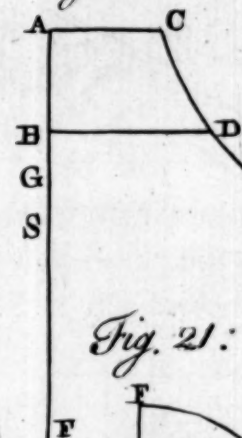


Fig. 19.

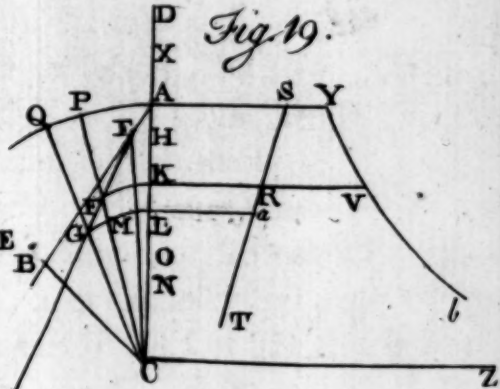


Fig. 17.

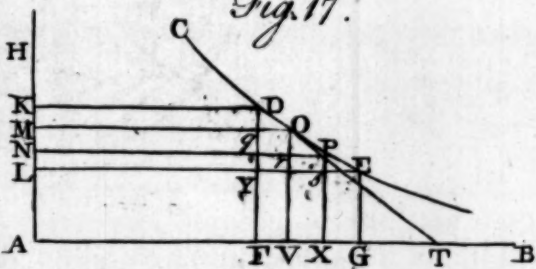


Fig. 21.

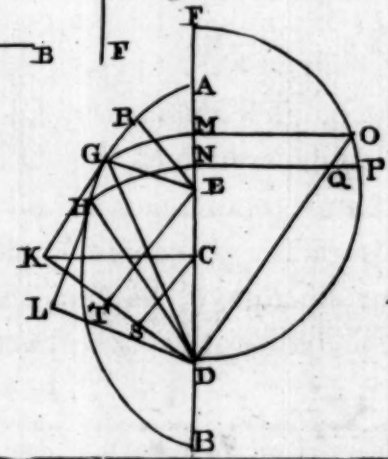


Fig. 22.

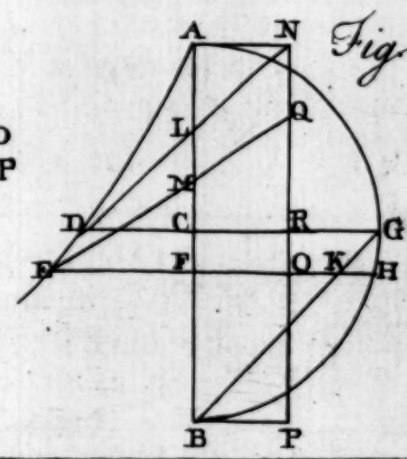
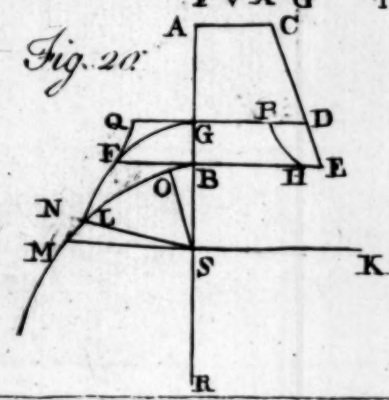


Fig. 20.



force at F is to the centripetal force at C as the square of BC to the square of BF.
Q. E. D.

P R O P. XXIV. Fig. 21.

If a body revolve in an ellipse, and be acted upon by a centripetal force tending to the focus of the ellipse; the centripetal force will be reciprocally as the square of the distance from the focus.

Let a body revolve in the ellipse whose transverse axis is AB, and acted upon by a centripetal force tending to the focus D; produce DA to F; and let DF, AB, be equal; draw DG, DH, to any two points G, H, in the ellipse; and draw DK, DL, perpendicular to GK, HL, tangents to the ellipse at G, H; with the centre D, and distances DG, DH, describe the circles GM, HN, meeting AB in M, N; draw MO, NP, perpendicular to AD, meeting the semicircle described upon DF in O, P; join DO meeting NP in Q.

Suppose a body to descend or ascend in

H

the

the line AD, and at the point A to have acquired a velocity equal to the velocity the body describing the ellipse has at the point A; the velocity therefore of the bodies at G and M will [13.] be equal; and likewise the velocities of the bodies at H and N will be equal: but the velocity of the body describing the ellipse at the point H is [12.] to the velocity at the point G as DK to DL; therefore the velocity of the body at N is to the velocity at M as DK to DL: but [21.] DK is to DL as PN to NQ; therefore the velocity of the body at N is to the velocity at M as PN to NQ; and therefore [23.] the centripetal force at N will be to the centripetal force at M as the square of DM to the square of DN; that is, as the square of DG to the square of DH: but because DH, DN, are equal, the centripetal force at H will be equal to the centripetal force at N; and because DG, DM, are equal, the centripetal force at G will be equal to the centripetal force at M; therefore the centripetal force at H will be to the centripetal force at G as the square of DG to the square of DH.

OTHER

OTHERWISE, *Fig. 23.*

Let S be the focus of the ellipse, AB the greater axis, C the centre, CD half the lesser axis; and suppose the body, in an indefinitely little part of time, to describe the part AE ; let F be any other point in the ellipse, and FG a part described by the body in a time equal to that in which the part AE was described; join SE , SF , SG ; draw EK perpendicular to AB , meeting AB in K , and GM perpendicular to SF , meeting SF in M ; let FP be a tangent to the ellipse at the point F ; draw GP parallel to SF , meeting FP in P ; join CF meeting the ellipse in the point H ; and let GN , drawn parallel to FP , meet SF , CF , in N , O ; draw CQ parallel to FP , meeting the ellipse in Q , and SF in V ; complete the parallelogram $FCQT$; and draw FR perpendicular to CQ , meeting CQ in R .

Because QV , GN , are parallel, the angles RVF , GNM , will be equal; and the angles FRV , GMN , are likewise equal, because both right; therefore the triangles

H 2

FRV ,

FRV, GMN, will be similar; therefore MG will be to GN as RF to FV: but [Con. 5. 25.] FV is equal to CA, therefore MG is to GN as RF to CA: but [Con. 2. 20.] the parallelogram FCQT is equal to the rectangle ACD; therefore RF is to CA as DC to CQ; therefore MG is to GN as DC to CQ: but GN, GO, may be considered as equal, because FG is described in an indefinitely little part of time; therefore MG is to GO as DC to CQ; therefore the square of MG is to the square of GO as the square of DC to the square of CQ: but [Con. 2. 15.] the square of GO is to the rectangle HOF as the square of CQ to the square of CF; therefore, by equality, the square of GM is to the rectangle HOF as the square of DC to the square of CF: but the rectangles HOF, HFO, may be considered as equal; and therefore the square of GM will be to the rectangle HFO as the square of DC to the square of CF: but the rectangle HFO is to the rectangle HFN as FO to FN; that is, as CF to FV or CA; that is, as the square of CF to the rectangle FCA; therefore

therefore the square of GM is to the rectangle HFN as the square of DC to the rectangle FCA : but the rectangle HFN is to the rectangle contained by AB , FN , as HF to AB ; that is, as FC to CA ; that is, as the rectangle FCA to the square of CA ; therefore the square of GM is to the rectangle contained by AB , FN , as the square of CD to the square of CA ; that is, [Con. 2. 6.], as the square of EK to the rectangle BKA : but the rectangles BKA , BAK , may be considered as equal; therefore the square of GM is to the rectangle contained by AB , FN , as the square of EK to the rectangle BAK ; therefore, alternating, the square of GM will be to the square of EK as the rectangle contained by AB , FN , to the rectangle BAK ; that is, as FN to AK : but because AE , FG , are described in equal times, the triangles SAE , SFG , will be equal; therefore the rectangle contained by SA , EK , will be equal to the rectangle contained by SF , GM ; therefore GM will be to EK as SA to SF ; therefore the square of GM will be to the square of EK as the square of SA to the square

square of SF ; therefore FN is to AK as the square of SA to the square of SF : but because FN , GP , are equal, the centripetal force at F will be to the centripetal force at A as FN to AK ; therefore the centripetal force at F is to the centripetal force at A as the square of SA to the square of SF . And therefore the centripetal force is reciprocally as the square of the distance from the focus. *Q. E. D.*

P R O P. XXV. *Fig. 24.*

If a body revolve in an ellipse, it is required to find the law of the centripetal force tending to the centre of the ellipse.

Let AB be the greater axis, C the centre, and CD half the lesser axis; and suppose the body, in an indefinitely little part of time, to describe the part AE ; and let F be any other point in the ellipse, and FG a part of the ellipse, described in a time equal to that in which the part AE was described; join CE , CF , CG ; draw EK perpendicular to AB , meeting AB in K ; and GM
perpendicular

perpendicular to CF, meeting CF in M; let FP be a tangent to the ellipse at the point F; draw GP parallel to CF, meeting FP in P; let GN, drawn parallel to PF, meet CF in N; draw CQ parallel to FP, meeting the ellipse in Q; complete the parallelogram FCQT; draw FR perpendicular to CQ, meeting CQ in R; and let CF meet the ellipse in H.

Because the parts AE, FG, are described in equal times, the triangles ACE, FCG, will be equal; therefore the rectangle contained by CA, EK, will be equal to the rectangle contained by CF, GM; therefore EK is to GM as CF to CA; therefore the square of EK is to the square of GM as the square of CF to the square of CA: because GN is parallel to CQ, the angles GNM, FCR, will be equal; and the angles GMN, FRC, are equal, because both right; therefore the triangles GNM, FCR, are equiangular; therefore GM is to GN as FR to FC; therefore the square of GM is to the square of GN as the square of FR to the square of FC: because the square of EK is

to

to the square of GM as the square of CF to the square of CA; and the square of GM is to the square of GN as the square of FR to the square of CF; therefore, by perturbate equality, the square of EK is to the square of GN as the square of FR to the square of CA: because [Con. 2. 20.] the parallelogram FCQT is equal to the rectangle DCA, FR is to CA as CD to CQ; therefore the square of FR is to the square of CA as the square of CD to the square of CQ; therefore the square of EK is to the square of GN as the square of CD to the square of CQ: but [Con. 2. 15.] the square of GN is to the rectangle HNF as the square of CQ to the square of CF; therefore the square of EK is to the rectangle HNF as the square of CD to the square of CF; therefore, inversely, the rectangle HNF is to the square of EK as the square of CF to the square of CD: but the square of EK is [Con. 2. 6.] to the rectangle BKA as the square of CD to the square of CA; therefore, by equality, the rectangle HNF is to the rectangle BKA as the square of CF to the square of CA: but because FG, AE, are indefinitely little,
the

the rectangles HNF, HFN, may be considered as equal; and likewise the rectangles BKA, BAK, may be considered as equal; therefore the rectangle HFN is to the rectangle BAK as the square of CF to the square of CA; therefore the rectangle CFN is to the rectangle CAK as the square of CF to the square of CA; therefore FN is to AK as CF to CA: but FN, GP, are equal; therefore GP is to AK as CF to CA: but GP is to AK as the centripetal force at F to the centripetal force at A; therefore the centripetal force at F is to the centripetal force at A as CF to CA. And therefore the centripetal force tending to the centre of the ellipse is as the distance from the centre of the ellipse. Q. E. I.

P R O P. XXVI. Fig. 25.

Let there be an ellipse whose transverse axis is AB; let F be a given point in AB; suppose a body acted upon by a centripetal force, tending to the point F, to describe the ellipse: it is required to determine the law of the centripetal force tending to the point F.

I

Let

Let C be the centre of the ellipse, and E the focus; and let CD be half the lesser axis; let KL be a part of the ellipse described in an indefinitely little part of time; join FK , CK , EK , and FL ; draw LP perpendicular to FK , meeting FK in P ; draw LM parallel to FK , meeting KM , a tangent to the ellipse at K , in M ; and draw LN parallel to KM , meeting FK , CK , in N , O ; draw CQ parallel to KM , meeting FK , EK , in Q , X ; and draw FY parallel to CQ or KM , meeting EK in Y ; let CK , CQ , meet the ellipse in T , R ; and draw KS perpendicular to CR , meeting CR , FY , in S , Z ; let AG be a part of the ellipse described in a time equal to that in which the part LK was described; draw GH perpendicular to AB , meeting AB in H ; join FG ; and let the rectangle contained by CA , KY , be equal to the rectangle AFV .

Because LN , FZ , are parallel, the angles LNP , KFZ , will be equal; and the angles LPN , KZF , are likewise equal, because both right; therefore the triangles LPN , FKZ , are similar; therefore NL will
be

be to LP as FK to KZ; therefore the rectangle contained by NL, KZ, will be equal to the rectangle contained by LP, FK: but because KL, AG, are described in equal times, the triangles FKL, FAG, will be equal; therefore the rectangle contained by LP, FK will be equal to the rectangle contained by GH, FA; therefore the rectangle contained by LN, KZ, will be equal to the rectangle contained by GH, AF; therefore LN will be to GH as AF to KZ; that is, as the rectangle contained by AF, CR, to the rectangle contained by KZ, CR: but because KZ is to KY as KS to KX or CA; that is, as CD to CR; the rectangle contained by KZ, CR, will be equal to the rectangle contained by CD, KY; therefore LN will be to GH as the rectangle contained by AF, CR, to the rectangle contained by CD, KY; therefore the square of LN will be to the square of GH, in the duplicate ratio of the rectangle contained by AF, CR, to the rectangle contained by CD, KY: but the square of GH is to the rectangle BHA in the duplicate ratio of CD to CA; that is, in the du-

plicate ratio of the rectangle contained by
 CD, KY, to the rectangle contained by
 CA, KY; therefore the square of LN will
 be to the rectangle BHA, in the duplicate
 ratio of the rectangle contained by AF, CR,
 to the rectangle contained by CA, KY;
 but because the rectangle contained by CA,
 KY, is equal to the rectangle AFV, the
 rectangle contained by AF, CR, will be to
 the rectangle contained by CA, KY, as the
 rectangle contained by AF, CR, to the
 rectangle AFV; that is, as CR to FV;
 therefore the square of LN will be to the
 rectangle BHA, in the duplicate ratio of
 CR to FV; that is, as the square of CR to
 the square of FV; therefore, inversely, the
 rectangle BHA will be to the square of LN
 as the square of FV to the square of CR:
 but because KL is indefinitely little, LN,
 LO, may be considered as equal; there-
 fore the rectangle BHA will be to the
 square of LO as the square of FV to the
 square of CR: but the square of LO is to
 the rectangle TOK as the square of CR to
 the square of CK; therefore, by equality,
 the rectangle BHA will be to the rectangle
 TOK

TOK as the square of FV to the square of CK: but because AG, KL, are indefinitely little, the rectangles BHA, BAH, may be considered as equal; and likewise the rectangles TOK, TKO, may be considered as equal; therefore the rectangle BAH will be to the rectangle TKO as the square of FV to the square of CK: but the rectangle TKO is to the rectangle TKN as KO to KN; that is, as CK to KQ; that is, as the square of CK to the rectangle CKQ; therefore the rectangle BAH will be to the rectangle TKN as the square of FV to the rectangle CKQ: but the rectangle TKN is to the rectangle contained by AB, KN, as KC to CA; that is, as the rectangle CKQ to the rectangle contained by CA, KQ; therefore the rectangle BAH will be to the rectangle contained by AB, KN, as the square of FV to the rectangle contained by AC, KQ; therefore AH will be to KN as the square of FV to the rectangle contained by AC, KQ.

Because the rectangle AFV is equal to the rectangle contained by CA, KY, FV will be to CA as KY to AF; therefore the
square

square of FV will be to the square of CA as
 the square of KY to the square of AF; that
 is, as the parallelopiped whose base is the
 square of KY, and altitude FK, to the pa-
 rallelopiped whose base is the square of AF,
 and altitude FK; but the square of CA is
 to the rectangle contained by AC, KQ,
 as AC to KQ; that is, as XK to KQ;
 that is, as YK to KF; that is, as the cube
 of YK to the parallelopiped whose base is
 the square of KY, and altitude FK; there-
 fore, by perturbate equality, the square of FV
 will be to the rectangle contained by AC, KQ
 as the cube of KY to the parallelopiped whose
 base is the square of AF, and altitude FK;
 therefore AH will be to KN as the cube of
 KY to the parallelopiped whose base is the
 square of AF, and altitude FK: but be-
 cause GH is parallel to the tangent to the
 ellipse at the point A, and LN parallel to
 the tangent to the ellipse at the point K;
 and AG, LK, are indefinitely little, and
 described in equal times; the centripetal
 force at A will be to the centripetal force at
 K as AH to KN; therefore the centripetal
 force at A is to the centripetal force at K as
 the

the cube of KY to the parallelopiped whose base is the square of AF , and altitude FK .

Q. *E. D.*

COR. This proposition comprehends the two former. For if the point F coincide with the focus E , the point Y will coincide with the point E ; and the cube of KY will be equal to the cube of KE ; likewise the parallelopiped whose base is the square of AF , and altitude FK , will be equal to the parallelopiped whose base is the square of AE , and altitude KE ; therefore the centripetal force at A is to the centripetal force at K as the cube of KE to the parallelopiped whose base is the square of AE , and altitude EK ; that is, as the square of KE to the square of AE ; which agrees with prop. 24.

Again, if the point F coincide with the centre C , KY will be equal to CA ; therefore the cube of KY will be equal to the cube of CA ; and the parallelopiped whose base is the square of AF , and altitude FK , will be equal to the parallelopiped whose base is the square of AC , and altitude CK ; therefore

therefore the centripetal force at A is to the centripetal force at K as the cube of AC to the parallelopiped whose base is the square of AC, and altitude CK; that is, as AC to CK; which agrees with prop. 25.

P R O P. XXVII. *Fig. 26.*

If a body describe an hyperbola, it is required to find the law of the centripetal force tending to the focus of the hyperbola.

Let S be the focus of the hyperbola, AB the transverse axis, C the centre, and CD half the second axis; and suppose the body, in an indefinitely little part of time, to describe the part AE; and let F be any other point in the hyperbola, and FG a part described by the body in a time equal to that in which the part AE was described; join SE, SF, SG; draw EK perpendicular to SA, meeting SA in K, and GM perpendicular to SF, meeting SF in M; let FP be a tangent to the hyperbola at the point F; draw GP parallel to SF, meeting FP in P; join CF, and produce CF to H, so that
CF,

CF, CH, may be equal; and let GN, drawn parallel to FP, meet SF, CF, in N, O; let CQ the semidiameter conjugate to CF, meet SF in V; complete the parallelogram FCQT; and draw FR perpendicular to CQ, meeting CQ in R.

Because QV, GN, are parallel, the angles RVF, GNM, will be equal; and the angles FRV, GMN, are likewise equal, because both right; therefore the triangles FRV, GMN, will be similar; therefore GM will be to GN as FR to FV: but [Con. 5. 25.] FV is equal to CA; therefore MG is to GN as FR to CA; but [Con. 3. 45.] the parallelogram FCQT is equal to the rectangle ACD; therefore FR is to CA as CD to CQ; therefore MG is to GN as DC to CQ; but GN, GO, may be considered as equal, because FG is described in an indefinitely little part of time; therefore MG is to GO as DC to CQ; therefore the square of MG is to the square of GO as the square of DC to the square of CQ: but [Con. 3. 28.] the square of GO is to the rectangle HOF as the square of CQ to the

K square

square of CF ; therefore, by equality, the square of GM is to the rectangle HOF as the square of CD to the square of CF : but the rectangles HOF , HFO , may be considered as equal; and therefore the square of GM will be to the rectangle HFO as the square of CD to the square of CF : but the rectangle HFO is to the rectangle HFN as FO to FN ; that is, as CF to FV or CA ; that is, as the square of CF to the rectangle FCA ; therefore the square of GM is to the rectangle HFN as the square of CD to the rectangle FCA : but the rectangle HFN is to the rectangle contained by AB , FN , as HF to AB ; that is, as FC to CA ; that is, as the rectangle FCA to the square of CA ; therefore the square of GM is to the rectangle contained by AB , FN , as the square of CD to the square of CA : but [Con. 3. 7.] the square of CD is to the square of CA as the square of EK to the rectangle BKA ; therefore the square of GM is to the rectangle contained by AB , FN , as the square of EK to the rectangle BKA : but the rectangles BKA , BAK , may be considered as equal; therefore the
 square

square of GM is to the rectangle contained by AB, FN, as the square of EK to the rectangle BAK; therefore, alternating, the square of GM will be to the square of EK as the rectangle contained by AB, FN, to the rectangle BAK; that is, as FN to AK: but because AE, FG, are described in equal times, the triangles SAE, SFG, will be equal; therefore the rectangle contained by SA, EK, will be equal to the rectangle contained by SF, GM; therefore GM will be to EK as AS to SF; therefore the square of GM will be to the square of EK as the square of AS to the square of SF: but because FN, GP, are equal, the centripetal force at F will be to the centripetal force at A as FN to AK; that is, as the square of GM to the square of EK; that is, as the square of AS to the square of SF; and therefore the centripetal force is reciprocally as the square of the distance from the focus S. Q. E. L

K 2 L E M

L E M M A. *Fig. 27.*

Let there be a parabola whose focus is S, and vertex A; join SA; draw SF to any point F, in the parabola; and draw SQ perpendicular to FQ, a tangent to the parabola at F, meeting FQ in Q: FS will be to SQ as SQ to SA.

Let FQ meet AS in R; join AQ; draw FT perpendicular to AS, meeting AS in T; and draw FC parallel to AS.

Because FR is a tangent to the parabola at F, the angle SFQ will [Con. 1. 9.] be equal to the angle QFC; that is, equal to the angle SRQ; and the angles SQF, SQR, are equal, because both right; and SQ is common; therefore FQ, QR, are equal: likewise the angles FSQ, QSR, are equal: and because TA is [Con. 1. 10.] equal to AR, therefore FQ is to QR as TA to AR; therefore FT, AQ, are parallel; therefore the angle SAQ is equal to the angle STF; that is, equal to the angle SQF: and because the angles FSQ, QSR,

QSR, are equal, the triangles FSQ, QSA, are similar; therefore FS is to SQ as SQ to SA. Q. E. D.

PROP. XXVIII. Fig. 27.

If a body describe a parabola, it is required to find the law of the centripetal force tending to the focus of the parabola.

Let S be the focus of the parabola, AB the axis, and A the vertex; and suppose the body, in an indefinitely little part of time, to describe the part AE; and let F be any other point in the parabola, and FG a part described by the body in a time equal to that in which the part AE was described; join SE, SF, SG; draw EK perpendicular to SA, meeting SA in K, and GM perpendicular to SF, meeting SF in M; let FP a tangent to the parabola at F, meet SA in R; draw FC parallel to AB, and GP parallel to SF, meeting FP in P, and GN parallel to FP, meeting SF, FC, in N, O; and draw SQ perpendicular to FR, meeting FR in Q.

Because

Because GN , FQ , are parallel, the angles GNM , SFQ , will be equal, and the angles GMN , SQF , likewise are equal, because both right; therefore the triangles GNM , SFQ , will be similar; therefore NG will be to GM as FS to SQ ; but GN , GO , may be considered as equal, because FG is described in an indefinitely little part of time; therefore OG is to GM as FS to SQ ; therefore the square of OG will be to the square of GM as the square of FS to the square of SQ ; but [lem.] FS is to SQ as SQ to SA ; therefore the square of FS is to the square of SQ as FS to SA ; therefore the square of OG will be to the square of GM as FS to SA . Again, because the square of GO is [Con. 1. 13.] equal to the rectangle contained by FO and the parameter of the diameter CF ; and four times SF is equal to the parameter of the diameter CF ; the square of GO will be equal to four times the rectangle SFO ; therefore four times the rectangle SFO will be to the square of GM as FS to SA ; that is, as four times the rectangle SFO to four times the

the rectangle contained by SA, FO; therefore the square of GM is equal to four times the rectangle contained by SA, FO: but [Con. 1. 12.] the square of EK is equal to four times the rectangle SAK; therefore the square of GM will be to the square of EK as four times the rectangle contained by SA, FO, to four times the rectangle SAK; that is, as FO to AK: but because NO, FR, are parallel, and likewise FO, SR, parallel, the triangles FNO, FSR, will be similar; therefore FN is to FO as FS to SR: but FS, SR, are equal; therefore FN, FO, likewise are equal; therefore the square of GM is to the square of EK as FN to AK: but because AE, FG, are described in equal times, the triangles SAE, SFG, will be equal; therefore the rectangle contained by SA, EK, will be equal to the rectangle contained by SF, GM; therefore GM will be to EK as AS to SF; therefore the square of GM will be to the square of EK as the square of AS to the square of SF: but the square of GM is to the square of EK as FN to AK; therefore FN is to AK as the square

square of AS to the square of SF : but because FN, GP, are equal, the centripetal force at F is to the centripetal force at A as FN to AK ; therefore the centripetal force at F is to the centripetal force at A as the square of AS to the square of SF ; and therefore the centripetal force is reciprocally as the square of the distance from the focus S.

Q. E. I.

TRACT

T R A C T II.

P R O P. I. *Fig. 28.*

LET S be a given point, and ABC , ADE , two curves, so related to each other, that joining SA , and drawing any two equal lines SB , SD , to the curves ABC , ADE , the angle ASB is to the angle ASD in a given ratio; suppose that of p to q : let BF , DG , be tangents to the curves ABC , ADE , at B , D ; and draw SF , SG , perpendicular to BF , DG , meeting BF , DG , in F , G : the rectangle contained by SF , DG , will be to the rectangle contained by SG , BF , as p to q .

Draw SH meeting the curve ABC in the point H , indefinitely near to the point B ; and let SK , equal to SH , meet the curve ADE in the point K , indefinitely near to the point D ; draw HL , KM , perpendicular to SB , SD , meeting SB , SD , in L , M .
Because the triangles BHL , BSF , are
L similar,

fimilar, BL will be to LH as BF to FS;
 therefore the rectangle contained by BF,
 HL, will be equal to the rectangle contain-
 ed by SF, BL. The same way it is shown,
 that the rectangle contained by DG, KM,
 is equal to the rectangle contained by SG,
 DM; therefore the rectangle contained by
 DG, KM, will be to the rectangle contain-
 ed by BF, HL, as the rectangle contained
 by SG, DM, to the rectangle contained
 by SF, BL: but because BS, SD, are e-
 qual, and likewise HS, SK, equal; there-
 fore BL, DM, are equal; and therefore
 the rectangle contained by SG, DM, will
 be to the rectangle contained by SF, BL,
 as SG to SF; that is, as the rectangle con-
 tained by SG, BF, to the rectangle SFB;
 therefore the rectangle contained by DG,
 KM, will be to the rectangle contained by
 BF, HL, as the rectangle contained by
 SG, BF, to the rectangle SFB; and, alter-
 nating, the rectangle contained by DG,
 KM, will be to the rectangle contained by
 SG, BF, as the rectangle contained by BF,
 HL, to the rectangle SFB; that is, as HL
 to SF; that is, as the rectangle contained
 by

by HL, DG, to the rectangle contained by SF, DG; therefore, inversely, the rectangle contained by SG, BF, will be to the rectangle contained by DG, KM, as the rectangle contained by SF, DG, to the rectangle contained by HL, DG; therefore, alternating, the rectangle contained by SG, BF, will be to the rectangle contained by SF, DG, as the rectangle contained by DG, KM, to the rectangle contained by HL, DG; that is, as KM to HL; and therefore, inversely, the rectangle contained by SF, DG, will be to the rectangle contained by SG, BF, as HL to KM. Because the angle ASH is to the angle ASK as p is to q ; that is, as the angle ASB to the angle ASD; the angle BSH will be to the angle DSK as p to q ; and therefore HL will be to KM as p is to q ; and therefore the rectangle contained by SF, DG, will be to the rectangle contained by SG, BF, as p is to q . Q. E. D.

PROP. II. Fig. 29.

Let there be an ellipse, whose transverse axis is AB, and focus S; in SA take SF, equal to SB; and let H be a point in FS produced; in AF produced take FG equal to AH; and upon GH let there be a semicircle GOH described; let there be a curve ADE, so related to the ellipse, that if, with the centre S, and any distance SC, a circle be described meeting the ellipse in C, and the curve ADE in D; and SC, SD, be drawn; the angle ASC is to the angle ASD, in the subduplicate ratio of the rectangle ASF to the rectangle AHF: draw SD, SE, to any two points D, E, in the curve ADE; and draw SK, SL, perpendicular to DK, EL, tangents to the curve ADE, at D, E, meeting DK, EL, in K, L; with the centre S, and distances SD, SE, describe the circles DM, EN, meeting AS in M, N; draw MO, NP, perpendicular to AS, meeting the semicircle GOH in O, P; and join SO, meeting NP in Q: KS will be to SL as PN to NQ.

Let

Let R be the other focus of the ellipse; and join RC ; draw SV perpendicular to CV , a tangent to the ellipse at C , and meeting CV in V ; and in SA take SX equal to AB .

Because the square of SV is [cor. prop. 21. Tr. 1.] to the rectangle ASB as SC to CR ; that is, as SM to MX ; that is, as the square of SM to the rectangle SMX ; alternating, the square of SV will be to the square of SM or SC as the rectangle ASB or SAX to the rectangle SMX ; therefore, inversely, the square of SC will be to the square of SV as the rectangle SMX to the rectangle SAX ; therefore, by division, the square of CV will be to the square of SV as the rectangle AMF to the rectangle ASB ; therefore CV will be to VS in the subduplicate ratio of the rectangle AMF to the rectangle ASB : but the rectangle contained by CV , SK , is to the rectangle VSK as CV to VS ; therefore the rectangle contained by CV , SK , is to the rectangle VSK , in the subduplicate ratio of the rectangle AMF to the rectangle ASB : and because the rectangle contained by SV , DK ,

is

is to the rectangle contained by CV, SK, [1.] in the subduplicate ratio of the rectangle ASB to the rectangle AHF; and the rectangle contained by CV, SK, is to the rectangle VSK in the subduplicate ratio of the rectangle AMF to the rectangle ASB; therefore, by perturbate equality, the rectangle contained by SV, DK, is to the rectangle VSK, in the subduplicate ratio of the rectangle AMF to the rectangle AHF: but the rectangle contained by SV, DK, is to the rectangle VSK as DK to SK; therefore DK is to KS in the subduplicate ratio of the rectangle AMF to the rectangle AHF; therefore the square of DK is to the square of KS as the rectangle AMF to the rectangle AHF; therefore, by composition, the square of DS is to the square of SK as the rectangle GMH to the rectangle AHF; and, alternating, the square of DS will be to the rectangle GMH as the square of SK to the rectangle AHF: but because GOH is a semicircle, the square of MO is equal to the rectangle GMH; therefore the square of DS or SM is to the square of MO as the square of SK to the rectangle AHF; that is,

is, the square of SK is to the rectangle AHF as the square of SM to the square of MO. The same way it is shown, that the square of SL is to the rectangle AHF as the square of SN to the square of NP; therefore, inversely, the rectangle AHF will be to the square of SL as the square of NP to the square of NS: but because the square of SK is to the rectangle AHF as the square of SM to the square of MO; that is, as the square of SN to the square of NQ; and the rectangle AHF is to the square of SL as the square of PN to the square of NS; therefore, by perturbate equality, the square of KS will be to the square of SL as the square of PN to the square of NQ; and therefore KS is to SL as PN to NQ.
 Q. E. D.

PROP. III. Fig. 30.

Let there be a semicircle whose diameter is AB; and let S be a given point in AB; let there be a curve ADE, such, that if from any two points D, E, in the curve ADE, there be drawn DF, EG, perpendicular to AB, meeting

meeting AB in F, G, and the semicircle in H, K; and joining SH, meeting GK in L; EG is to FD as the square of KG to the square of GL; let DM a tangent to the curve ADE at the point D, meet AB in M; and let SH meet the circle in N; SF will be to FM as the rectangle SHN to the square of HF.

From O, a point in the curve ADE, indefinitely near to the point D, draw OP perpendicular to AB, meeting AB in P, the circle in Q, T, and SH in R; and draw DV parallel to AB, meeting OP in V.

Because the rectangle OPV is to the square of FD as OP to PV or FD; that is, from the nature of the curve ADE, as the square of QP to the square of PR; the rectangle OPV will be to the square of QP as the square of FD to the square of PR; therefore the rectangle OVP will be to the rectangle TRQ or HRN as the rectangle OPV to the square of QP: but because QQ is indefinitely near to DH, the rectangle OPV may be considered as equal to the square of DF,

Plate. 4.

Fig. 23.

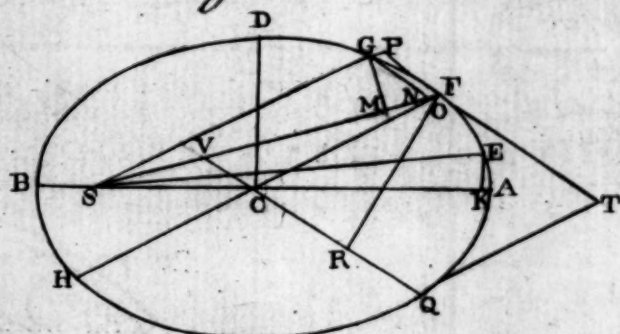


Fig. 24.

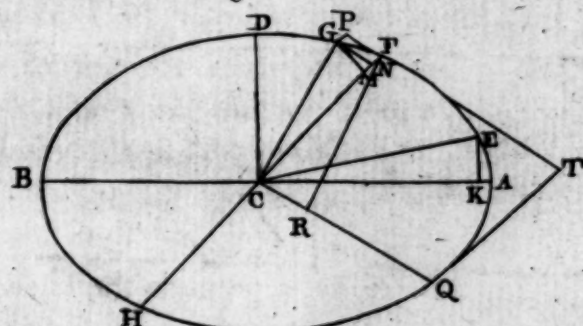


Fig. 25.

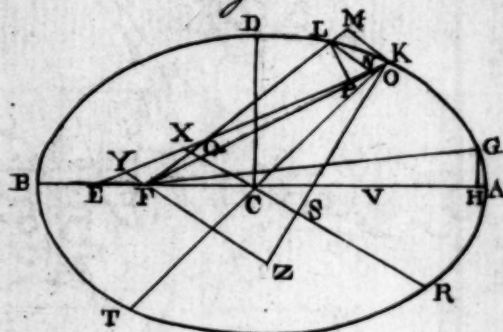


Fig. 26.

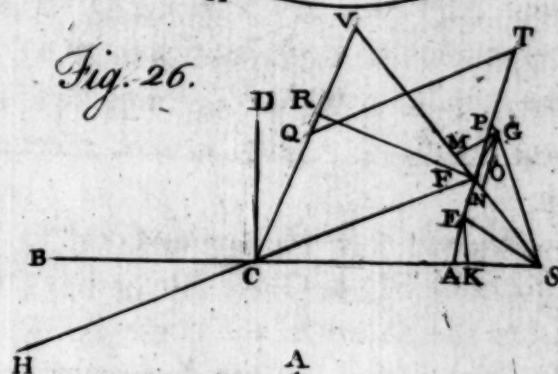


Fig. 27.

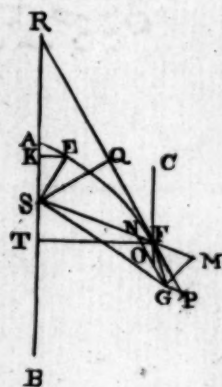
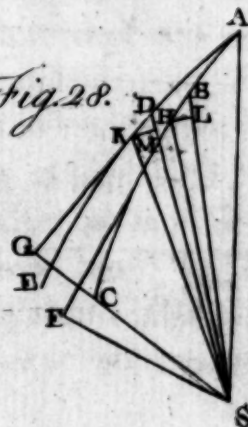


Fig. 28.



DF, the square of PQ may be considered as equal to the square of FH, and the rectangles NRH, NHR, may likewise be considered as equal; therefore the rectangle OVP will be to the rectangle NHR as the square of DF to the square of FH. Again, because the triangles OVD, DFM, may be considered as similar, DV or FP will be to OV as MF to FD; but the rectangle DFP is to the rectangle OVP as FP to OV; and the rectangle MFD is to the square of FD as MF to FD; therefore the rectangle DFP is to the rectangle OVP as the rectangle MFD to the square of FD; and because the rectangle OVP is to the rectangle NHR as the square of FD to the square of FH; therefore, by equality, the rectangle DFP will be to the rectangle NHR as the rectangle MFD to the square of FH. Again, because the rectangle DFP is to the rectangle contained by DF, HR, as FP to HR; that is, as FS to SH; that is, as the rectangle DFS to the rectangle contained by DF, SH; and the rectangle contained by DF, HR, is to the rectangle NHR as DF

M

to

to HN; that is, as the rectangle contained by DF, SH, to the rectangle SHN; the rectangle DFP will be to the rectangle NHR as the rectangle DFS to the rectangle SHN: but the rectangle DFP is to the rectangle NHR as the rectangle MFD to the square of FH; therefore the rectangle DFS is to the rectangle SHN as the rectangle MFD to the square of FH; therefore, alternating, the rectangle DFS is to the rectangle MFD as the rectangle SHN to the square of FH: but the rectangle DFS is to the rectangle MFD as SF to FM; therefore SF is to FM as the rectangle SHN to the square of FH. *Q. E. D.*

P R O P. IV. *Fig. 31.*

Let A, B, be two points in CD, the semidiameter of a circle whose centre is C; and let the rectangle ACB be equal to the square of CD; from any point E, in the circle, draw EF perpendicular to CD, meeting CD in F; and draw AG perpendicular to EG, a tangent to the circle at E, meeting

ing EG in G: AG will be to BF as AC to CD.

Join CE, DE; and let DE meet AG in H; draw DK parallel to AG or CE, meeting EG in K; let DC meet the circle in L; and join EL.

Because EK is a tangent to the circle at E; the angle DEK will be equal to the angle DLE; that is, equal to the angle DEF; and the angles DKE, DFE, are equal, because both right; therefore (because DE is common to both the triangles DKE, DFE) DF will be equal to DK; therefore DF is to GH as DK to GH; that is, as DE to EH; that is, as DC to CA: but because the rectangle ACB is equal to the square of CD, BC will be to CD as CD to CA; therefore BD is to DA as CD to CA; therefore DF is to GH as BD to DA; therefore (because DA is equal to AH) BF is to AG as BD to DA; that is, as DC to CA; and therefore, inversely, AG is to BF as AC to CD. *Q. E. D.*

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PROP.

PROP. V. Fig. 32.

In the semidiameter of a circle whose centre is C, let there be taken two points A, B; and let the rectangle ACB be equal to the square of the semidiameter; draw AE meeting the circle in E, F; and draw EG perpendicular to AB, meeting AB in G; the rectangle AEF will be equal to twice the rectangle contained by AC, BG.

Join CE, meeting the circle in D; and join FD; and let AH, drawn perpendicular to EH, a tangent to the circle at E, meet EH in H.

Because EH is a tangent to the circle at E, the angle AEH will be equal to the angle EDF; and the angles AHE, EFD, are equal, because both right; therefore the triangles AEH, EFD, are equiangular; therefore EA is to AH as DE to EF; therefore the rectangle AEF is equal to the rectangle contained by AH, DE; that is, equal to twice the rectangle contained by AH, CE; but because [4.] AH is to BG

as

as AC to CE, the rectangle contained by AH, CE, will be equal to the rectangle contained by AC, BG; therefore twice the rectangle contained by AH, CE, will be equal to twice the rectangle contained by AC, BG; therefore the rectangle AEF will be equal to twice the rectangle contained by AC, BG. $\mathcal{Q}$. E. D.

COR. From A draw AL to any point L in the circle, meeting the circle again in M; and draw LN perpendicular to AB, meeting AB in N; the rectangle AEF will be to the rectangle ALM as GB to BN.

Because the rectangle AEF is equal to twice the rectangle contained by AC, BG, and the rectangle ALM equal to twice the rectangle contained by AC, BN, the rectangle AEF will be to the rectangle ALM as twice the rectangle contained by AC, BG, to twice the rectangle contained by AC, BN; that is, as GB to BN. $\mathcal{Q}$. E. D.

PROP.

PROP. VI. Fig. 33.

Let there be a semicircle whose diameter is AB, and centre C; let S be a given point in AB; in CB take CQ, on the same side the centre C, with the point S; and let the rectangle QCS be equal to the square of CB; let DEF, KLM, be two curves, such, that if from any two points G, H, in AS, there be drawn GE, HF, perpendicular to AS, meeting the curve DEF in E, F, and likewise meeting the curve KLM in L, M; the square of HS is to the square of SG as GE to HF; likewise the rectangle QGE is to the rectangle QHF as the rectangle SGL to the rectangle SHM: let GE, HF, meet the semicircle in N, O; join SN, meeting OH in Z: suppose a body to descend or ascend in the line AS; and the velocity at H to be to the velocity at G as OH to HZ; the centripetal force at H will be to the centripetal force at G as HM to GL.

Let APR be a curve, meeting GN, HO,
in

in P, R, so related to the semicircle, that HR is to GP as the square of OH to the square of HZ; and let PT, RV, tangents to the curve APR at P, R, meet AS in T, V; let NS, SO, meet the circle in X, Y.

Because the rectangle QGE is to the rectangle contained by QH, GE, as QG to QH; that is, [cor. to prop. 5.] as the rectangle SNX to the rectangle SOY; that is, as the parallelopiped whose base is the rectangle SNX, and altitude GE, to the parallelopiped whose base is the rectangle SOY, and altitude GE; and the rectangle contained by QH, GE, is to the rectangle QHF as GE to HF; that is, as the parallelopiped whose base is the rectangle SOY, and altitude GE, to the parallelopiped whose base is the rectangle SOY, and altitude HF; therefore, by equality, the rectangle QGE will be to the rectangle QHF as the parallelopiped whose base is the rectangle SNX, and altitude GE, to the parallelopiped whose base is the rectangle SOY, and altitude HF: but the rectangle SGL is to the rectangle SHM as the rectangle

rectangle QGE to the rectangle QHF; therefore the rectangle SGL is to the rectangle SHM as the parallelopiped whose base is the rectangle SNX, and altitude GE, to the parallelopiped whose base is the rectangle SOY, and altitude HF: but the rectangle SHM is to the rectangle MHV as SH to HV; that is, as the rectangle [3.] SOY to the square of OH; that is, as the parallelopiped whose base is the rectangle SOY, and altitude HF, to the parallelopiped whose base is the square of OH, and altitude HF; therefore the rectangle SGL is to the rectangle MHV as the parallelopiped whose base is the rectangle SNX, and altitude GE, to the parallelopiped whose base is the square of OH, and altitude HF; therefore, inversely, the rectangle MHV will be to the rectangle SGL as the parallelopiped whose base is the square of OH, and altitude HF, to the parallelopiped whose base is the rectangle SNX, and altitude GE: but the rectangle SGL is to the rectangle LGT as SG to GT; that is, as the rectangle SNX to the square of GN; that is, as the parallelopiped whose base is the

the

the rectangle SNX, and altitude GE, to the parallelopiped whose base is the square of GN, and altitude GE; therefore the rectangle MHV will be to the rectangle LGT as the parallelopiped whose base is the square of OH, and altitude HF, to the parallelopiped whose base is the square of NG, and altitude GE: but because HF is to GE as the square of GS to the square of SH; that is, as the square of GN to the square of HZ; the parallelopiped whose base is the square of GN, and altitude GE, will be equal to the parallelopiped whose base is the square of HZ, and altitude HF; therefore the rectangle MHV will be to the rectangle LGT as the parallelopiped whose base is the square of OH, and altitude HF, to the parallelopiped whose base is the square of HZ, and altitude HF; that is, as the square of OH to the square of HZ: but RH is to PG as the square of OH to the square of HZ; therefore the rectangle MHV will be to the rectangle LGT as RH to GP; that is, as the rectangle contained by RH, GT, to the rectangle PGT; therefore, alternating, the rectangle MHV

N

will

will be to the rectangle contained by RH, GT, as the rectangle LGT to the rectangle PGT; that is, as LG to GP; that is, as the rectangle contained by LG, HV, to the rectangle contained by GP, HV; and therefore, alternating, the rectangle MHV will be to the rectangle contained by LG, HV, as the rectangle contained by RH, GT, to the rectangle contained by GP, HV; and therefore MH will be to GL as the rectangle contained by RH, GT, to the rectangle contained by GP, HV.

Because the square of OH is to the square of HZ as RH to GP, OH will be to HZ in the subduplicate ratio of RH to GP: but the velocity at H is to the velocity at G as OH to HZ; therefore the velocity at H is to the velocity at G, in the subduplicate ratio of RH to GP: but [5. Tr. 1.] the centripetal force at H is to the centripetal force at G as the rectangle contained by RH, GT, to the rectangle contained by GP, HV; therefore the centripetal force at H is to the centripetal force at G as MH to GL. Q. E. D.

PROP.

P R O P. VII. Fig. 34.

Let there be an ellipse whose transverse axis is AB, and focus S; in SA take SF, equal to SB; and let H be a point in AS produced; in SA produced take FG equal to AH; upon GH let there be a semicircle described; bisect AF in K; in KH take KL on the same side the point K with the point S, and let the rectangle LKS be equal to the square of KH; let there be two curves DCE, MNO, such, that if from any two points P, Q, in GS, there be drawn PC, QE, perpendicular to SA, meeting the curve DCE in C, E, and likewise meeting the curve MNO in N, O; the square of QS is to the square of SP as PC to QE; likewise the rectangle LPC is to the rectangle LQE as the rectangle SPN to the rectangle SQO: let ART be a curve so related to the ellipse, that if with the centre S, and any distance SV, a circle be described meeting the ellipse in V, and the curve ART in R; and SV, SR, be drawn; the angle ASV is to the angle ASR

in the subduplicate ratio of the rectangle ASF to the rectangle AHF: let a body acted upon by a centripetal force tending to the point S, describe the curve ART; with the centre S, and distances SR, ST, describe the circles RP, TQ, meeting the curve ART in R, T, and SA in P, Q; draw PN, QO, perpendicular to AS, meeting the curve MNO in N, O: the centripetal force at R will be to the centripetal force at T as PN to QO.

Let PN, QO, meet the curve DCE in C, E, and the semicircle in X, Y; and join SX, meeting QY in Z; and draw Sa, Sb, perpendicular to Ra, Tb, tangents to the curve ART in R, T, meeting Ra, Tb, in a, b; and join ST.

Suppose a body to descend or ascend in the line AS, and at the point A to have acquired a velocity equal to the velocity the body describing the curve ART has at the point A; the velocities, therefore, of the bodies at R and P will [13. Tr. 1.] be equal: but the velocity of the body describing the curve ART at the point T is [12. Tr. 1.]

to

to the velocity at the point R as Sa to Sb ; therefore the velocity at Q is to the velocity at P as Sa to Sb : but Sa is to Sb [2.] as YQ to QZ ; therefore the velocity at Q is to the velocity at P as YQ to QZ ; therefore [6.] the centripetal force at P will be to the centripetal force at Q as PN to QQ : but because SP , SR , are equal, the centripetal forces at P and R will be equal: and because SQ , ST , are equal, the centripetal forces at Q and T are equal: therefore the centripetal force at R is to the centripetal force at T as PN to QQ . *Q. E. D.*

P R O P. VIII. *Fig. 35.*

In the line AB take a point C, such, that AC is double of CB; and take any point D in AB: the ratio of CB to BD will be greater than the ratio of the square of AD to the square of AC.

Produce AB to E, so that AE may be equal to AC; and bisect EB in F.

1st, If the point D be between C, B; because FD is greater than FC, the rectangle

angle EDB will be less than the rectangle ECB; that is, less than the square of AC; therefore the ratio of the rectangle EDC to the rectangle EDB will be greater than the ratio of the rectangle EDC to the square of AC; that is, the ratio of CD to DB will be greater than the ratio of the rectangle EDC to the square of AC; therefore, by composition, the ratio of CB to BD will be greater than the ratio of the square of AD to the square of AC.

2d, If the point D be between A, C; because FD is less than FC, the rectangle EDB will be greater than the rectangle ECB; that is, greater than the square of AC; therefore the ratio of the rectangle EDB to the rectangle EDC will be greater than the ratio of the square of AC to the rectangle EDC; that is, the ratio of BD to DC will be greater than the ratio of the square of AC to the rectangle EDC; and, by division, the ratio of BC to CD will be greater than the ratio of the square of AD to the rectangle EDC; therefore, inversely, the ratio of DC to CB will be less than the ratio of the rectangle EDC to the square of

of

of AD; and, by composition, the ratio of DB to BC will be less than the ratio of the square of AC to the square of AD; therefore, inversely, the ratio of CB to BD will be greater than the ratio of the square of AD to the square of AC. *Q. E. D.*

P R O P. IX. *Fig. 36.*

Let A be a given point in the straight line AB; and CDE a curve, such, that if from any two points C, D, in the curve CDE, there be drawn CF, DG, perpendicular to AB, meeting AB in F, G, the square of AG will be to the square of AF as CF to DG; in AF take the point H, such, that AH may be triple of HF; join CH: CH will be a tangent to the curve CDE at C.

From any point K, in CH, draw KG perpendicular to AB, meeting AB in G, and the curve CDE in D.

Because AH is triple of HF, AF will be double of FH; therefore [8.] the ratio of FH to HG will be greater than the ratio of the

the square of AG to the square of AF : but because CF is to KG as FH to HG , and CF to DG as the square of AG to the square of AF ; therefore the ratio of CF to KG will be greater than the ratio of CF to DG ; therefore DG is greater than KG ; therefore the point K is without the curve CDE . The same way it is shown, that any other point in CH is without the curve CDE ; and therefore CH is a tangent to the curve CDE at the point C . *Q. E. D.*

P R O P. X. *Fig. 37.*

Let A be a given point in the straight line AB ; and CDE a curve, such, that if from any two points D, E , in the curve CDE , there be drawn DF, EG , perpendicular to AB , meeting AB in F, G , DF will be to EG as the square of AG to the square of AF ; draw AH perpendicular to AB : if from any two points D, E , in the curve, there be drawn DK, EL , perpendicular to AH , meeting AH in K, L ; and likewise DF, EG , perpendicular to AB , meeting AB in F, G ; the space $DKLE$

Fig. 29.

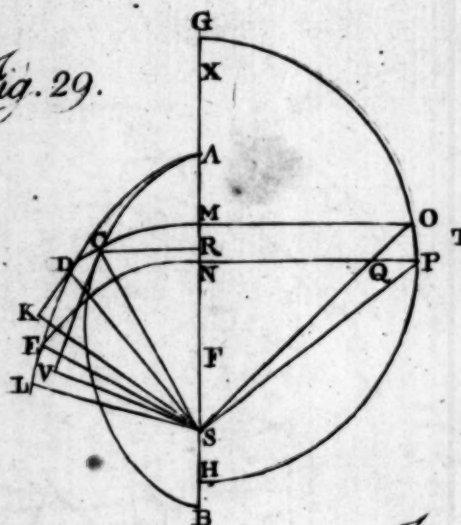


Fig. 30.

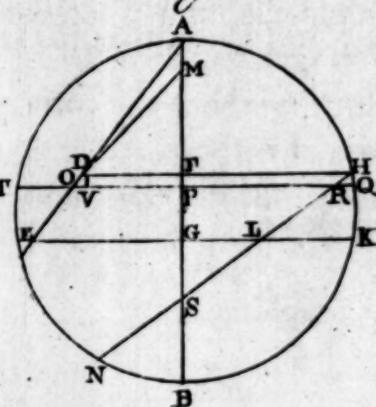


Fig. 31.

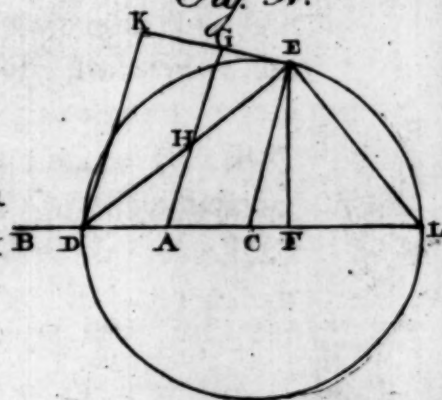


Fig. 32.

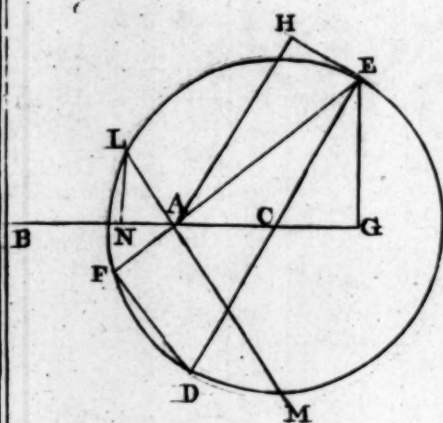


Fig. 33.

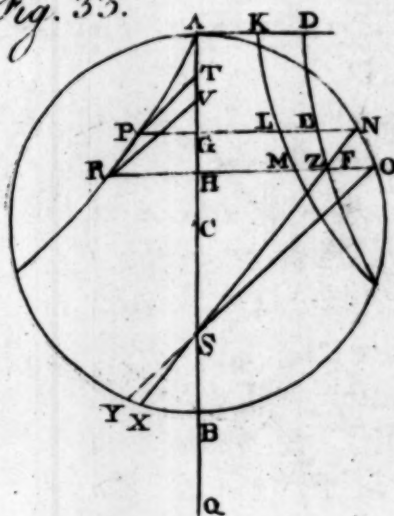
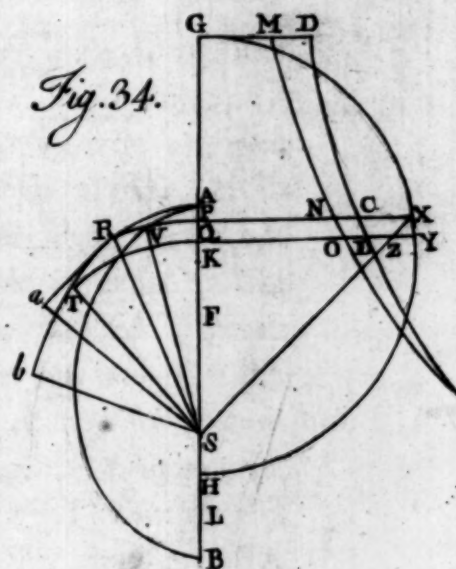


Fig. 34.



DKLE will be double of the space DFGE; likewise the space DFGE will be equal to the excess of the rectangle AFD above the rectangle AGE.

Suppose KL to be divided into indefinitely little parts in M, N, &c.; draw MO, NP, &c. parallel to DK or EL, meeting the curve CDE in O, P, &c.; draw OV, PX, &c. parallel to DF or EG, meeting AB in V, X, &c.; let DF meet OM in q , OV meet NP in r , and PX meet EL in s ; take, in AB, a point T, such, that AT may be triple of TF; and join DT.

Because AT is triple of TF, DT will [9.] be a tangent to the curve CDE at the point D; and because MO is indefinitely near to DK, the triangles DFT, DqO , may be considered as similar; therefore DF is to FT as Dq to qO ; therefore the rectangle contained by Dq , FT, is equal to the rectangle contained by Oq , DF; that is, equal to the rectangle FO: but because AT is triple of TF, AF will be double of FT; therefore the rectangle contained by AF,

O

Dq ,

Dq , will be double of the rectangle contained by FT , Dq ; and therefore the rectangle DM will be double of the rectangle FO . The same way it is shown, that the rectangle ON is double of the rectangle VP , and the rectangle PL double of the rectangle XE , and so on. But the sum of the rectangles DM , ON , PL , &c. may be considered as making up the figure $DKLE$; and the sum of the rectangles FO , VP , XE , &c. may be considered as making up the figure $DFGE$; therefore the figure $DKLE$ is double of the figure $DFGE$.

Again, let DF meet EL in Y . Because the figure $DKLE$ is double of the figure $DFGE$, the rectangle DL will be equal to the figure $DFGE$, together with the rectangle FE : let the rectangle FL be added to both, and the rectangle AD will be equal to the figure $DFGE$, together with the rectangle AE ; and therefore the space $DFGE$ will be equal to the excess of the rectangle AD above the rectangle AE ; that is, equal to the excess of the rectangle AFD above the rectangle AGE . Q. E. D.

P R O P.

PROP. XI. Fig. 38.

Let S be a given point in the straight line AS ,
and CDE a curve, such, that if from any
two points A, B , in AS , there be drawn
 AC, BD , perpendicular to AS , meeting
the curve CDE in C, D ; the square of AS
is to the square of SB as BD to AC : the
rectangle SAC will be to the area $CABD$
as SB to BA ; likewise the rectangle SBD
will be to the area $CABD$ as SA to AB .

Because the rectangle SBD is to the rect-
angle contained by SB, AC , as BD to AC ;
that is, as the square of AS to the square of
 SB ; and the rectangle contained by $SB,$
 AC , is to the rectangle SAC as BS to SA ;
that is, as the square of SB to the rectangle
 ASB ; therefore, by equality, the rectangle
 SBD will be to the rectangle SAC as the
square of AS to the rectangle ASB ; that is,
as AS to SB ; therefore, by division, the
excess of the rectangle SBD above the rect-
angle SAC will be to the rectangle SAC as
 AB to BS ; therefore, inversely, the rect-
angle

angle SAC will be to the excess of the rectangle SBD above the rectangle SAC as SB to BA; but the area CABD is [10.] equal to the excess of the rectangle SBD above the rectangle SAC; therefore the rectangle SAC is to the area CABD as SB to BA.

Again, because the rectangle SBD is to the rectangle SAC as AS to SB, and the rectangle SAC is to the area CABD as SB to BA, therefore the rectangle SBD will be to the area CABD as SA to AB. Q. E. D.

P R O P. XII. Fig. 39.

Suppose a body acted upon by a centripetal force tending to the centre S, to descend in the straight line AS, and the centripetal force to be reciprocally as the square of the distance from the centre S; upon AS describe a semicircle AFS; from any two points B, G, in AS, draw BF, GH, perpendicular to AS, meeting the semicircle AFS, in F, H; and join SF, meeting GH in K: suppose the body to fall from the point A, the velocity at G will be to the velocity at B as HG to GK.

Let

Let CDE be a curve, such, that if from any two points B, G, in AS, there be drawn BD, GE, perpendicular to AS, meeting the curve CDE in D, E, the square of BS is to the square of SG as GE to BD; and draw AC perpendicular to AS, meeting the curve CDE in C; let BD, CE, meet the semicircle described upon AS in F, H; and join SF, meeting GH in K.

Because the centripetal force at G is to the centripetal force at B as the square of BS to the square of SG, and the square of BS is to the square of SG as GE to BD, the centripetal force at G will be to the centripetal force at B as GE to BD; and therefore [3. Tr. 1.] the velocity at G is to the velocity at B in the subduplicate ratio of the area CAGE to the area CABD: but because the rectangle SAC is [11.] to the area CAGE as SG to GA; that is, because of the circle, as the square of SG to the square of GH; inversely, the area CAGE will be to the rectangle SAC as the square of HG to the square of GS: but the rectangle SAC is to the area CABD as SB to BA; that is, as the square of SB to the square of BF; that is,

is, as the square of SG to the square of GK ; therefore, by equality, the area $CAGE$ is to the area $CABD$ as the square of HG to the square of GK ; therefore HG is to GK in the subduplicate ratio of the area $CAGE$ to the area $CABD$: but the velocity at G is to the velocity at B in the subduplicate ratio of the area $CAGE$ to the area $CABD$; therefore the velocity at G is to the velocity at B as HG to GK . Q. E. D.

P R O P. XIII. Fig. 40.

Suppose a body acted upon by a centripetal force tending to the centre S , to descend in the straight line AS , and the centripetal force to be reciprocally as the square of the distance from the centre S ; let the body, at the point B , be projected in the direction BC , perpendicular to AB , with the velocity that would be acquired in falling from A to B : the centrifugal force at the point B , arising from the projectile force in the direction BC , will be to the centripetal force at the point B as twice AB to AS .

Let

Let DEF be a curve, such, that if from any two points A, B, in AS, there be drawn AD, BE, perpendicular to AS, meeting the curve DEF in D, E, the square of AS is to the square of SB as BE to AD.

Because the centripetal force at B is to the centripetal force at A as the square of AS to the square of SB, the centripetal force at B will be to the centripetal force at A as BE to AD; therefore [II. Tr. I.] the centrifugal force at B, arising from the projectile force in the direction BC, perpendicular to AB, will be to the centripetal force at the point B as twice the area DABE to the rectangle SBE: but because the rectangle SBE is to the area DABE [II.] as SA to AB, the rectangle SBE will be to twice the area DABE as SA to twice AB; therefore, inversely, twice the area DABE will be to the rectangle SBE as twice AB to AS; and therefore the centrifugal force at B, arising from the projectile force in the direction BC, perpendicular to AB, will be to the centripetal force at the point B as twice AB to AS. Q. E. D.

P R O P.

PROP. XIV. Fig. 41.

Let there be a semicircle whose diameter is AB , and centre C , and let there be a curve BDE , so related to the semicircle, that if from any point F in the semicircle there be drawn FG perpendicular to AB , meeting AB in G , and the curve BDE in H , BG will be to GF as GH to AC ; draw AN perpendicular to AB ; it is evident that AN is an asymptote to the curve BDE : join BF , the space contained by AG , GH , the curve HE indefinitely produced, and its asymptote AN , is double of the sector ABF ; likewise the space contained by AB , and the curve BDE indefinitely produced, and its asymptote AN , is double of the semicircle AFB .

Suppose the arc FA to be divided into indefinitely little parts in a , b , c , d , &c. draw ae , bf , cg , db , &c. perpendicular to AB , meeting AB in e , f , g , h , &c. and the curve BDE in k , l , m , n , &c.; join Ba , Bb , Bc , Bd , &c.; let FK be a tangent to

to the semicircle at F; draw BK perpendicular to FK, meeting FK in K; join CF, FA; and draw ao parallel to AB, meeting FG in o .

Because the angle KFB is equal to the angle FAB; that is, equal to the angle BFG; and the angles BKF, BGF, are likewise equal, because both right; the triangles BKF, BGF, will be every way equal, (because BF is common to both); therefore BK, BG, are equal: but, because of the curve BDE, BG is to GF as HG to AC or CF; therefore BK is to GF as HG to CF; therefore, alternating, BK is to GH as GF to FC: but GF is to FC as ao to aF ; that is, as Ge to aF ; therefore BK is to GH as Ge to aF ; and therefore the rectangle HGe is equal to the rectangle contained by BK, Fa: but the rectangle contained by BK, Fa, is double of the triangle FBa; therefore the space He is double of the triangle FBa. The same way it is shown, that the spaces kf , lg , mb , &c., are double of the triangles aBb , bBc , cBd , &c.; therefore the space contained by AG, GH, the curve HE indefinitely produced, and its

F asymptote

asymptote AN, is double of the sector ABF. From this it is evident, that the space contained by AB, and the curve BDE indefinitely produced, and its asymptote, is double of the semicircle AFB. *Q. E. D.*

P R O P. XV. *Fig. 42.*

Supposing a body to descend in the straight line AS, from the point A, and the centripetal force tending to the centre S, to be reciprocally as the square of the distance from the centre; upon AS let there be a semicircle ADS described; from any two points B, C, in AS let there be drawn BD, CE, perpendicular to AS, meeting the semicircle in D, E; and join DS, SE: the time of descent from A to B will be to the time of descent from A to C as the sector ASD to the sector ASE; and the time of descent from A to B will be to the time of descent from A to S as the sector ASD to the semicircle ADS.

Bisect AS in F; and let SGH be a curve, such, that if from any point D in the semicircle

circle there be drawn DB perpendicular to AS, meeting AS in B, and the curve SGH in G; SB is to BD as GB to AF; let EC meet the curve SGH in K; and let SD meet EC in M; draw AL perpendicular to AS, on the same side of AS that the curve SGH is.

Because, from the nature of the curve SGH, SC is to CE as CK to AF, inversely, EC will be to CS as AF to CK: but SC is to CM as SB to BD; that is, as GB to AF; therefore, by perturbate equality, EC is to CM as GB to CK: but the velocity at C is to the velocity at B [12.] as EC to CM; therefore the velocity at C is to the velocity at B as GB to CK; and therefore [4. Tr. 1.] the time of descent from A to B will be to the time of descent from A to C as the space contained by AB, BG, the curve GH indefinitely produced, and its asymptote AL, to the space contained by AC, CK, the curve KH indefinitely produced, and its asymptote AL: but the space contained by AB, BG, the curve GH indefinitely produced, and its asymptote AL, is [14.] double of the sector ASD; and the space contained by AC, CK, the curve KH indefinitely

indefinitely produced, and its asymptote AL, is double of the sector ASE; therefore the time of descent from A to B will be to the time of descent from A to C as the double of the sector ASD to the double of the sector ASE; that is, as the sector ASD to the sector ASE.

From this it is evident, that the time of descent from A to B is to the time of descent from A to S as the sector ASD to the semicircle ADS. Q. E. D.

P R O P. XVI. Fig. 43.

Supposing two bodies to descend from A and B in the straight lines AS, BS, to the centre S, and the centripetal force tending to the centre S to be reciprocally as the square of the distance from the centre; the square of the time of descent from A to S will be to the square of the time of descent from B to S as the cube of AS to the cube of SB.

Upon AS, BS, describe the semicircles ACS, BDS; and let AE, BF, be described in an indefinitely little part of time; draw
EC

EC perpendicular to AS, meeting the semicircle ACS in C; and draw FD perpendicular to SB, meeting the semicircle BDS in D; join CS, SD; draw BG meeting the semicircle BDS in G, so that the semicircle BDS may be to the sector BGS as the semicircle ACS to the sector ASC; and draw GH perpendicular to BS, meeting BS in H.

Because BF, AE, are described in the same time, and the time is supposed indefinitely little, BF will be to AE as the centripetal force at B to the centripetal force at A; that is, as the square of AS to the square of SB: but BF is to AE as the rectangle SBF to the rectangle contained by SB, AE; therefore the rectangle SBF is to the rectangle contained by SB, AE, as the square of AS to the square of SB; therefore, alternating, the rectangle SBF will be to the square of AS as the rectangle contained by SB, AE, to the square of SB; that is, as AE to SB; that is, as the rectangle SAE to the rectangle ASB; therefore, alternating, the rectangle SBF will be to the rectangle SAE as the square of AS to the rectangle
angle

angle ASB; that is, as AS to SB. Because BF, AE, are indefinitely little, the rectangles SBF, SFB, may be considered as equal; and likewise the rectangles SAE, SEA, may be considered as equal; therefore the rectangle SFB will be to the rectangle SEA as AS to SB: but because of the semicircle, the rectangle SEA is equal to the square of EC; and the rectangle SFB equal to the square of FD; therefore the square of FD is to the square of EC as AS to SB; that is, as the cube of AS to the parallelopiped whose base is the square of AS, and altitude SB. Again, because the semicircle BDS is to the sector BGS as the semicircle ACS to the sector ASC; therefore the sectors ASC, GBS, are similar; therefore CE is to GH as AS to SB; and therefore the square of CE will be to the square of GH as the square of AS to the square of SB; that is, as the parallelopiped whose base is the square of AS, and altitude SB, to the cube of SB: but because the square of FD is to the square of EC as the cube of AS to the parallelopiped whose base is the square of

of

of AS, and altitude SB; and the square of EC is to the square of GH as the parallelopiped whose base is the square of AS, and altitude SB, to the cube of SB; therefore, by equality, the square of FD is to the square of GH as the cube of AS to the cube of SB. Because the time of descent from A to S is [15.] to the time of descent from A to E as the semicircle ACS to the sector ASC; that is, as the semicircle BDS to the sector GBS; and the time of descent from A to E is equal to the time of descent from B to F; therefore the time of descent from A to S is to the time of descent from B to F as the semicircle BDS to the sector GBS: but [15.] the time of descent from B to F is to the time of descent from B to S as the sector BSD to the semicircle BDS; therefore the time of descent from A to S is to the time of descent from B to S as the sector BSD to the sector GBS: and because the sectors BSD, GBS, are indefinitely little, the sector BSD will be to the sector GBS as DF to GH; therefore the time of descent from A to S is to the time of descent from B to S as DF to GH; and therefore

fore the square of the time of descent from A to S is to the square of the time of descent from B to S as the square of DF is to the square of GH; that is, as the cube of AS to the cube of SB.

P R O P. XVII. Fig. 44.

Let there be a circle whose diameter is AB, and centre S; and supposing a body to revolve in the circle, and the centripetal force tending to the centre S to be reciprocally as the square of the distance from the centre; produce SA to C; and let CS be equal to AB: the periodic time of the body revolving in the circle is double the time a body would take to descend from C to S.

Let AD be a part of the circle described in an indefinitely little part of time; join SD; and draw DG perpendicular to AB, meeting AB in G; let the body falling in the line CS, descend through the part CE in a time equal to that in which the part AD was described; draw FE perpendicular

to

to SC, meeting the semicircle described upon SC, in F; and join SF.

Because AD is described in an indefinitely little part of time, it is evident that AG would be described in the same time by the centripetal force tending to the centre S; therefore AG would be described in the same time that CE is described; therefore AG is to CE as the centripetal force at A to the centripetal force at C; that is, as the square of CS to the square of SA: but, because CS is double of SA, the square of CS will be quadruple the square of SA; therefore AG is quadruple of CE; and therefore the rectangle BAG will be quadruple the rectangle SCE: but, because AG, CE, are indefinitely little, the rectangles BAG, BGA, may be considered as equal; and likewise the rectangles SCE, SEC, may be considered as equal; therefore the rectangle BGA is quadruple the rectangle SEC: but, because of the circle, the rectangle BGA is equal to the square of DG; and the rectangle SEC is equal to the square of EF; therefore the square of DG is quadruple the square of FE; and therefore DG is double

Q

of

of FE: and because CS is double of SA, therefore DG is to FE as CS to SA; therefore the sectors ASD, CSF, are equal. Again, because the periodic time of the body revolving in the circle ADB is to the time in which the part AD was described as the circle ADB to the sector ASD, and the time in which the part AD was described is equal to the time in which a body would descend from C to E, and likewise the sector ASD is equal to the sector CSF; therefore the periodic time of the body revolving in the circle ADB is to the time in which the body would descend from C to E as the circle ADB to the sector CSF: but the time in which a body would descend from C to E is to the time in which a body would descend from C to S [15.] as the sector CSF to the semicircle CFS; therefore, by equality, the periodic time of the body revolving in the circle ADB is to the time in which a body would descend from C to S as the circle ADB to the semicircle CFS: but because AB, SC, are equal, therefore the circle ADB is double of the semicircle CFS; and therefore the periodic time

time of the body revolving in the circle ADB is double of the time a body would take to descend from C to S. Q. E. D.

P R O P. XVIII. Fig. 45.

Let there be an ellipse ABP whose greater axis is AP, and focus S; and supposing a body acted upon by a centripetal force tending to the focus S to revolve in the ellipse, produce SA to C; and let CS be equal to AP: the periodic time of the body revolving in the ellipse ABP is double of the time a body would take to descend from C to S.

Let D be the centre of the ellipse; and draw DB perpendicular to AP, meeting the ellipse in B, E; and upon SC describe the semicircle CHS; let AK be a part of the ellipse described in an indefinitely little part of time; join SK; and draw KN perpendicular to AP, meeting AP in N; let the body descending in the line CS descend through the part CL, in a time equal to that in which the part AK was described by the body revolving in the ellipse; draw

Q 2

LM

LM perpendicular to CS, meeting the femicircle in M; and join SM.

Because AK is described in an indefinitely little part of time, it is evident that AN would be described in the same time by the centripetal force tending to S; therefore AN, CL, would be described in equal times; therefore CL is to AN as the centripetal force at C to the centripetal force at A; that is, [24. Tr. I.], as the square of AS to the square of SC; but, because SC, AP, are equal, the rectangle SCL will be to the rectangle PAN as CL to AN; therefore the rectangle SCL is to the rectangle PAN as the square of AS to the square of SC: but, because CL, AN, are indefinitely little, the rectangles SCL, SLC, may be considered as equal; and likewise the rectangles PAN, PNA, may be considered as equal; therefore the rectangle SLC will be to the rectangle PNA as the square of AS to the square of SC: but [Con. 2. 6.] the rectangle PNA is to the square of KN as the square of AP or SC to the square of BE; therefore, by equality, the rectangle SLC will be to the square of KN as the square

square of AS to the square of BE: but, because of the semicircle, the square of ML is equal to the rectangle SLC; therefore the square of ML is to the square of KN as the square of SA to the square of BE; and therefore ML is to KN as SA to BE; therefore the rectangle contained by SA, KN, is equal to the rectangle contained by BE, ML; and therefore the rectangle contained by SC, ML, will be to the rectangle contained by SA, KN, as the rectangle contained by SC, ML, to the rectangle contained by BE, ML; that is, as SC or AP to BE: but, because CM, AK, are indefinitely little, the sector CSM will be to the sector ASK as the rectangle contained by SC, ML, to the rectangle contained by SA, KN; therefore the sector CSM is to the sector ASK as CS or AP to BE; that is, as the semicircle to the semiellipse; therefore, alternating, the sector CSM is to the semicircle as the sector ASK to the semiellipse; therefore, inversely, the semicircle is to the sector CSM as the semiellipse to the sector ASK: but the time of descent from C to S is to the time of descent

scant from C to L as the semicircle [15.] to the sector CSM; and the time in which the body revolves in the semiellipse is to the time in which the part AK is described as the semiellipse to the sector ASK; therefore the time in which the body revolves in the semiellipse is to the time in which the part AK is described as the time of descent from C to S to the time of descent from C to L; but the time of descent from C to L is equal to the time in which the part AK is described; therefore the time in which the body revolves in the semiellipse is equal to the time of descent from C to S: but, because the ellipse is double of the semiellipse, the periodic time of the body revolving in the ellipse will be double of the time in which the body revolves in the semiellipse; therefore the periodic time of the body revolving in the ellipse is double of the time of descent from C to S. Q. E. D.

P R O P. XIX.

Supposing two bodies to revolve in different ellipses about a common focus, acted upon by centripetal

centripetal forces tending to the common focus; the square of the periodic time in the one will be to the square of the periodic time in the other, as the cube of the transverse axis of the one to the cube of the transverse axis of the other.

Because the bodies revolving in the ellipses are acted upon by centripetal forces tending to the focus of the ellipses, therefore [24. Tr. 1.] the centripetal forces are reciprocally as the square of the distances from the focus; therefore the periodic time of each of these bodies [18.] revolving in the ellipses is double the time in which the bodies would descend to the focus from a height equal to the transverse axis of the ellipse: but [16.] the squares of the times the bodies would take to descend to the focus from different heights are as the cubes of those heights; therefore the squares of the periodic times of the bodies revolving in the ellipses about a common focus are as the cubes of the transverse axis of the ellipses. Q. E. D.

COR. I. Supposing two bodies to revolve
in

in different circles about a common centre, and the centripetal force tending to the centre to be reciprocally as the square of the distance from the centre; the square of the periodic time in the one will be to the square of the periodic time in the other, as the cube of the diameter of the one to the cube of the diameter of the other.

COR. 2. Supposing two bodies acted upon by a centripetal force that is reciprocally as the square of the distance from the centre of force, the one to revolve in an ellipse, and the other in a circle, and the centre of force to be the focus of the ellipse, and likewise the centre of the circle; the square of the periodic time of the body revolving in the ellipse will be to the square of the periodic time of the body revolving in the circle, as the cube of the transverse axis of the ellipse to the cube of the diameter of the circle.

PROP.

PROP. XXI. Fig. 46.

Let a body acted upon by a centripetal force that is directly as its distance from the centre S descend in the straight line AS from the point A ; with the centre S , and distance SA , describe a circle; from any point B in AS draw BD perpendicular to AS , meeting the circle in D : the velocity at S will be to the velocity at B as AS to BD .

Let the circle meet AS in C ; draw AE perpendicular to AS ; and let AE represent the centripetal force at the point A ; join SE ; and draw BF parallel to AE , meeting SE in F ; BF therefore will represent the centripetal force at the point B ; and therefore the velocity at S will be to the velocity at B [3. Tr. 1.] in the subduplicate ratio of the triangle ASE to the trapezium $ABFE$: but, because the triangle ASE is to the triangle BSF as the square of AS to the square of SB , conversely, the triangle ASE will be to the trapezium $ABFE$ as the square of AS to the rectangle ABC ; that is, as the

R
square

square of AS to the square of BD; therefore AS is to BD in the subduplicate ratio of the triangle ASE to the trapezium ABFE; and therefore the velocity at S is to the velocity at B as AS to BD.
Q. E. D.

P R O P. XXI. *Fig. 47.*

Let AB be a quadrant of a circle whose centre is C; join CA, CB; in CD, drawn perpendicular to AC, take any given point D; let DE be a curve, such, that if from any point F in AC there be drawn FG perpendicular to AC, meeting the circle in G, and the curve DHE in H, the rectangle GFH is equal to the rectangle BCD; draw AK perpendicular to AC: it is evident that AK will be an asymptote to the curve DE; the space contained by AF, FH, the curve HE indefinitely produced, and its asymptote AK, will be to the space contained by AC, CD, the curve DE indefinitely produced, and its asymptote AK, as the arc AG to the arc AGB.

Suppose

Suppose the arc AG to be divided into indefinitely little parts in L, M, N, &c.; draw Lo, Mp, Nq, &c. perpendicular to AF, meeting AF in o, p, q, &c. and the curve HE in O, P, Q, &c.; join CG; and draw Lr parallel to AC, meeting GF in r and CG in s.

Because the rectangle GFH is equal to the rectangle BCD; that is, equal to the rectangle GCD; FH will be to CD as CG to GF; that is, as sG to Gr; but because LG is indefinitely little, GL may be considered as coinciding with the tangent to the circle at G; therefore the angle LGs is right; therefore the triangles LGr, rGs are similar; therefore sG is to Gr as GL to Lr or Fo; therefore FH is to CD as GL to Fo; therefore the rectangle HFo is equal to the rectangle contained by GL, CD. The same way it is shown, that the rectangle Oop is equal to the rectangle contained by LM, CD; and the rectangle Ppq, equal to the rectangle contained by MN, CD, and so on: but the rectangles HFo, Oop, Ppq, &c. may be considered as equal to the space contained by AF, FH, the curve HE in-

definitely produced, and its asymptote AK; and the rectangles contained by GL, CD; LM, CD; MN, CD, &c.; may be considered as equal to the rectangle contained by the arc AG, and CD; therefore the space contained by AF, FH, the curve HE indefinitely produced, and its asymptote AK, is equal to the rectangle contained by the arc AG, and CD. The same way it is shown, that the space contained by FC, CD, the curve DH, and FH, is equal to the rectangle contained by the arc BG, and CD; therefore the space contained by AC, CD, the curve DE indefinitely produced, and its asymptote AK, will be equal to the rectangle contained by the arc AGB, and CD; and therefore the space contained by AF, FH, the curve HE indefinitely produced, and its asymptote AK, will be to the space contained by AC, CD, the curve DE indefinitely produced, and its asymptote AK, as the rectangle contained by the arc AG, and CD, to the rectangle contained by the arc AGB, and CD; that is, as the arc AG to the arc AGB. Q. E. D.

PROP.

PROP. XXII. Fig. 48.

Suppose a body acted upon by a centripetal force tending to the centre C to descend in the straight line AC ; and let the centripetal force be as the distance of the body from the centre C ; draw CB perpendicular to CA ; with the centre C , and distance CA , describe a circle meeting CB in B ; and let D be any point in AC ; draw DE perpendicular to AC , meeting the circle in E : the time of descent from A to D will be to the time of descent from A to C as the arc AE to the arc AEB .

In BC produced, take any point F ; let FG be a curve, such, that if from any point D , in AC , there be drawn DE perpendicular to AC , meeting the circle in E , and the curve in H , the rectangle EDH is equal to the rectangle BCF ; draw AK perpendicular to AC .

Because the rectangle EDH is equal to the rectangle BCF , ED will be to BC as CF to DH ; but the velocity at D is to the velocity

velocity at C [20.] as DE to BC; therefore the velocity at D is to the velocity at C as CF to DH; therefore [4. Tr. I.] the time of descent from A to D will be to the time of descent from A to C, as the space contained by AD, DH, the curve HG indefinitely produced, and its asymptote AK, to the space contained by AC, CF, the curve FG indefinitely produced, and its asymptote AK: but the space contained by AD, DH, the curve HG indefinitely produced, and its asymptote AK, is to the space contained by AC, CF, the curve FG indefinitely produced, and its asymptote AK [21.], as the arc AE to the arc AEB; therefore the time of descent from A to D will be to the time of descent from A to C as the arc AE to the arc AEB. Q. E. D.

P R O P. XXIII. Fig. 49.

Suppose two bodies acted upon by a centripetal force that is directly as the distances from the centre of force, to descend from A and B to the centre C; the time of descent from

A

A to C will be equal to the time of descent from B to C.

Let AF , BG , be spaces passed over by the bodies at A and B in an indefinitely little part of time; with the centre C , and distances CA , CB , let there be circles described meeting CA , CB , in D , E ; draw FH , GK , perpendicular to CA , CB , meeting the circles AHD , BKE , in H , K ; draw CL , CM , perpendicular to CA , CB , meeting the circles AHD , BKE , in L , M ; and join CH , CK .

Because AF , BG , are described in an indefinitely little part of time, AF will be to BG as the centripetal force at A to the centripetal force at B ; that is, as CA to CB ; that is, as DA to EB ; that is, DA will be to EB as AF to BG ; therefore, alternating, DA will be to AF as EB to BG ; therefore, by division, DF will be to FA as EG to GB ; therefore the rectangles DFA , EGB , are similar; therefore the rectangle DFA is to the rectangle EGB as the square of AF to the square of BG ; that is, as the square of CA to the square of BC : but, because of
the

velocity at C [20.] as DE to BC; therefore the velocity at D is to the velocity at C as CF to DH; therefore [4. Tr. 1.] the time of descent from A to D will be to the time of descent from A to C, as the space contained by AD, DH, the curve HG indefinitely produced, and its asymptote AK, to the space contained by AC, CF, the curve FG indefinitely produced, and its asymptote AK: but the space contained by AD, DH, the curve HG indefinitely produced, and its asymptote AK, is to the space contained by AC, CF, the curve FG indefinitely produced, and its asymptote AK [21.], as the arc AE to the arc AEB; therefore the time of descent from A to D will be to the time of descent from A to C as the arc AE to the arc AEB. *Q. E. D.*

P R O P. XXIII. *Fig. 49.*

Suppose two bodies acted upon by a centripetal force that is directly as the distances from the centre of force, to descend from A and B to the centre C; the time of descent from A

A to C will be equal to the time of descent from B to C.

Let AF , BG , be spaces passed over by the bodies at A and B in an indefinitely little part of time; with the centre C , and distances CA , CB , let there be circles described meeting CA , CB , in D , E ; draw FH , GK , perpendicular to CA , CB , meeting the circles AHD , BKE , in H , K ; draw CL , CM , perpendicular to CA , CB , meeting the circles AHD , BKE , in L , M ; and join CH , CK .

Because AF , BG , are described in an indefinitely little part of time, AF will be to BG as the centripetal force at A to the centripetal force at B ; that is, as CA to CB ; that is, as DA to EB ; that is, DA will be to EB as AF to BG ; therefore, alternating, DA will be to AF as EB to BG ; therefore, by division, DF will be to FA as EG to GB ; therefore the rectangles DFA , EGB , are similar; therefore the rectangle DFA is to the rectangle EGB as the square of AF to the square of BG ; that is, as the square of CA to the square of BC : but, because of
the

the circle, the rectangle DFA is equal to the square of FH, and the rectangle EGB equal to the square of GK; therefore the square of FH is to the square of GK as the square of CA to the square of CB; therefore FH is to GK as CA to CB; that is, as CH to CK; therefore, alternating, FH is to HC as GK to KC; therefore (because the angles CFH, CGK, are both right) the triangles FCH, GCK, are similar; and the angles ACH, BCK, will be equal; therefore the arcs AH, BK, are similar; therefore the arc AH will be to the arc AHL as the arc BK to the arc BKM; but the time of descent from A to F is [22.] to the time of descent from A to C as the arc AH to the arc AHL; and the time of descent from B to G is to the time of descent from B to C as the arc BK to the arc BKM; therefore the time of descent from A to F is to the time of descent from A to C as the time of descent from B to G to the time of descent from B to C: but the time of descent from A to F is equal to the time of descent from B to G; therefore the time of descent

descent from A to C is equal to the time of descent from B to C. Q. E. D.

P R O P. XXIV. Fig. 50.

Let there be an ellipse whose axes are AB, DE, and centre C; suppose a body acted upon by a centripetal force tending to the centre C to describe the ellipse: the periodic time of the body describing the ellipse will be quadruple of the time the body would descend from A to C.

With the centre C, and distance CA, describe a circle meeting DE in F, G; let AH be a part of the ellipse described in an indefinitely little part of time; draw HK perpendicular to AC, meeting AC in K, and the circle in L; and join CH, CL.

Because the time in which the arc AH of the ellipse is described is to the time in which the arc AHD is described, as the portion ACH of the ellipse to the portion ACD of the ellipse; and the portion ACH of the ellipse is to the portion ACD, as the sector ACL to the sector ACF; that is, as the arc AL to the arc

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ALF;

ALF; therefore the time in which the arc AH of the ellipse is described is to the time in which the arc AHD is described as the arc AL to the arc ALF: but because the body describing the ellipse is acted upon by a centripetal force tending to the centre C, therefore [25. Tr. 1.] the centripetal force will be as the distance of the body from the centre; therefore the time in which the body would descend from A to K will be [22.] to the time in which the body would descend from A to C as the arc AL to the arc ALF; and therefore the time in which the arc AH of the ellipse is described will be to the time in which the arc AHD is described, as the time in which the body would descend from A to K to the time in which the body would descend from A to C: but because AH is described in an indefinitely little part of time, and HK is parallel to the tangent to the ellipse at the point A; therefore the time in which the arc AH of the ellipse is described will be equal to the time in which the body would descend from A to K; therefore the time in which the arc AHD is described will be equal to the

the time in which the body would descend from A to C: but the periodic time of the body describing the ellipse is quadruple of the time in which the arc AHD is described; therefore the periodic time of the body describing the ellipse will be quadruple of the time of descent from A to C. Q. E. D.

COR. From this and the preceding proposition it is evident, that the periodic times of bodies revolving in different ellipses about a common centre C, and acted upon by centripetal forces tending to the common centre, will be equal.

PROP. XXV. Fig. 51.

Let A be a given point in the straight line AB; and let CD, FG, be two curves on the same side AB, so, that if from any two points H, K, in AB, there be drawn HC, KD, perpendicular to AB, meeting the curve CD in C, D, and likewise meeting the curve FG in F, G, the square of HA is to the square of AK as KD to HC, and likewise the cube of HA is to the cube of

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AK

AK as KG to HF; in HA take the point B, such, that HA is to AB as CH to HF; and draw BL perpendicular to AB, meeting one of the curves, suppose the curve CD, in L, the other curve FG will meet the curve CD in the point L; and if from the point L, where the two curves CD, FG, intersect each other, there be drawn LB perpendicular to AB, meeting AB in B; and from any point H in AB, there be drawn a line parallel to LB, meeting the curves CD, FG, in C, F; CH will be to HF as HA to AB.

Because BL is to HC as the square of HA to the square of AB, and the square of HA is to the square of AB as the cube of HA to the parallelopiped whose base is the square of AB, and altitude HA; BL will be to HC as the cube of HA to the parallelopiped whose base is the square of AB, and altitude HA: but because CH is to HF as HA to AB; and HA is to AB as the parallelopiped whose base is the square of BA, and altitude AH, to the cube of BA; therefore, by equality, BL is to HF as the cube

cube of AH to the cube of AB; and therefore it is evident that the point L is in the curve FG.

Again, because CH is to BL as the square of AB to the square of AH; and the square of AB is to the square of AH as the parallelopiped whose base is the square of AB, and altitude AH, to the cube of AH; CH will be to BL as the parallelopiped whose base is the square of BA, and altitude AH, to the cube of AH: but because of the curve FG, BL is to HF as the cube of AH to the cube of BA; therefore, by equality, CH is to HF as the parallelopiped whose base is the square of BA, and altitude AH, to the cube of BA; that is, as HA to AB.

Q. E. D.

Cor. If HA be to AB as CH to HF, and LB be joined, LB will be perpendicular to AB.

PROP. XXVI. Fig. 52.

In the straight line AB take two points C, D; and let AC be to CB as AD to DB; bisect AB

AB in E; the rectangle CED will be equal to the square of EB; likewise the square of CA will be to the square of AD as CE to ED.

Because AC is to CB as AD to DB, AC, together with CB, will be to CB as AB to BD; that is, twice EC will be to CB as twice EB to BD; therefore EC will be to CB as EB to BD; therefore, conversely, CE will be to EB as EB to ED; and therefore the rectangle CED is equal to the square of EB.

Again, because CE is to EB as EB to ED, CB will be to BD as CE to EB; therefore the square of CB will be to the square of BD as the square of CE to the square of EB; that is, as the square of CE to the rectangle CED; that is, as CE to ED: but because AC is to CB as AD to DB, alternating, AC will be to AD as CB to BD; therefore the square of CA is to the square of AD as the square of CB to the square of BD; that is, as CE to ED.

Q. E. D.

P R O P.

P R O P. XXVII. *Fig. 53.*

Let A be a given point in the straight line AB; and let CD, EF, be two curves on the same side AB, so related to the line AB, that if from any two points G, H, in AB, there be drawn GC, HD, perpendicular to AB, meeting the curve CD in C, D, and the curve EF in E, F, the square of GA is to the square of AH as HD to GC, and likewise the cube of GA is to the cube of AH as HF to GE; from K, the point where the two curves intersect each other, let fall KB perpendicular to AB, meeting AB in B; in AB take any two points G, H, so that AG may be to GB as AH to HB; and draw GC, HD, perpendicular to AB, meeting the curve CD in C, D, and the curve EF in E, F: the space CGHD will be equal to the space EGHF.

Bisect AB in L: because AG is to GB as AH to HB, and AB is bisected in L, GL will be to LH [26.] as the square of GA to the square of AH: but, because of the
curve

curve CD, the square of GA is to the square of AH as HD to GC; therefore GL is to LH as HD to GC; therefore the rectangle LHD is equal to the rectangle LGC; and therefore the excess of the rectangle AHD above the rectangle AGC is equal to the excess of the rectangle contained by AL, HD, above the rectangle contained by AL, GC; that is, because AB is bisected in L, equal to the rectangle contained by LB, and the excess of HD above GC; but the excess of the rectangle AHD above the rectangle AGC is equal [10.] to the figure CGHD; therefore the rectangle contained by LB, and the excess of HD above GC, is equal to the figure CGHD. Again, because [25.] DH is to HF as HA to AB, and CG to GE as GA to AB, the rectangle AHF will be equal to the rectangle contained by AB, HD; likewise the rectangle AGE will be equal to the rectangle contained by AB, GC; therefore the excess of the rectangle AHF above the rectangle AGE will be equal to the rectangle contained by AB and the excess of HD above GC; and therefore, because AB is bisected

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Plate 7

Fig. 45

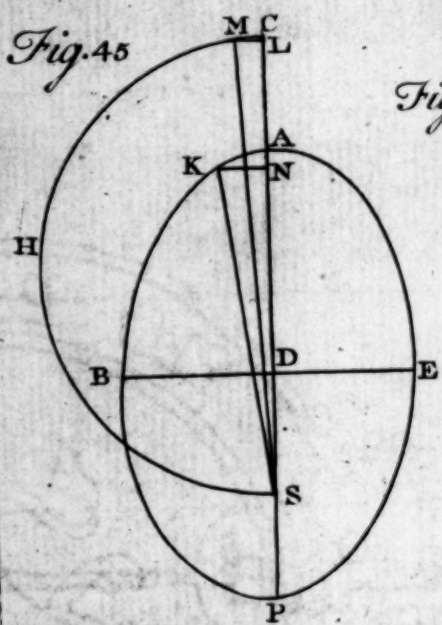


Fig 46

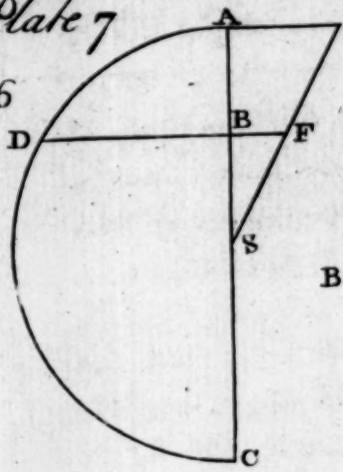


Fig 47

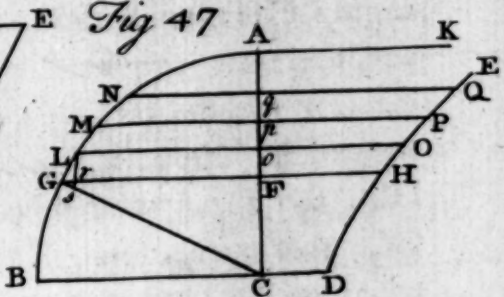


Fig 50

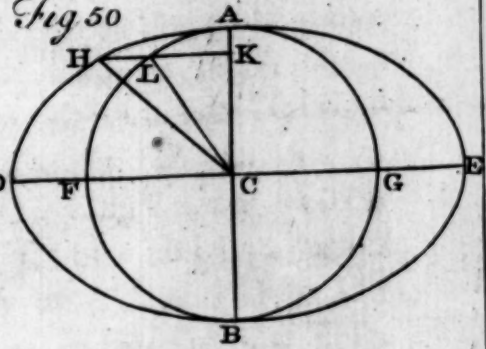


Fig 49

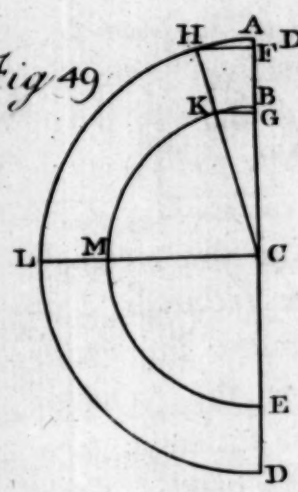


Fig 51

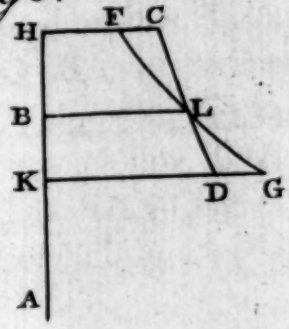


Fig 48

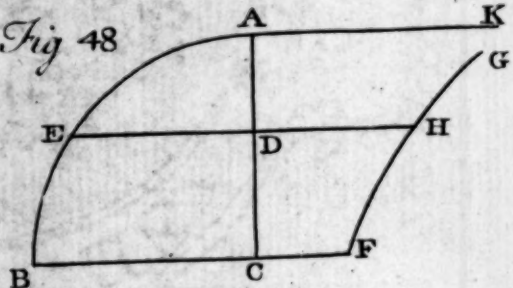
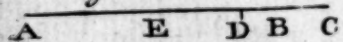


Fig 52



ed in L , the excess of the rectangle AHF above the rectangle AGE will be double of the rectangle contained by LB and the excess of HD above GC ; but the excess of the rectangle AHF above the rectangle AGE is [17. Tr. 1.] double of the space $EGHF$; therefore the rectangle contained by LB , and the excess of HD above GC , is equal to the figure $EGHF$; therefore the space $CGHD$ is equal to the space $EGHF$.
 $\square$ $E. D.$

COR. If from any point M between G , H , there be drawn MN perpendicular to GH , meeting the curves CD , EF , in N , O ; the space $CGMN$ will be greater than the space $EGMO$; likewise the space $FHMO$ will be greater than the space $DHMN$.

If the point M be between the points B , G , it is evident, that the space $CGMN$ is greater than the space $EGMO$. Let the point M be between the points B , H : because the space $CGHD$ is equal to the space $EGHF$, take away the common space $KEGHD$ from both, and the space EKC
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will be equal to the space DKF; but the space DKF is greater than the space KNO; therefore the space EKC is greater than the space KNO; let the space KEGMN be added to both, and the space CGMN will be greater than the space EGMO. *ai* *HOA*
 Again, because the space CGHD is equal to the space EGHF, and the space CGMN greater than the space EGMO; therefore the space FHMO will be greater than the space DHMN. *Q. E. D.* *C. A. S.*

PRO P. XXVIII. *Fig. 54.*

Let there be a straight line AS; and suppose a body at the point A to be acted upon by two forces, a centripetal and centrifugal force, with respect to the centre S, the centripetal to be reciprocally as the square of the distance from the centre, and the centrifugal to be reciprocally as the cube of the distance from the centre; at the point A let the centripetal be greater than the centrifugal; in AS take SB so, that AS may be to SB as the centripetal to the centrifugal force at the point A; in SB take SC so, that *SC*

SC may be to CB as SA to AB; a body acted upon by the above forces will descend from A to C, and no lower.

From A draw AD perpendicular to AS, and in AD take AD, AH, so, that DA may be to AH as AS to SB; and let DKE, HKF, be two curves so related to AS, that if from any two points A, C, in AS, there be drawn AD, CE, perpendicular to AS, meeting the curves DKE, HKF, in D, E, and H, F, the square of AS is to the square of SC as CE to AD, likewise the cube of AS is to the cube of SC as CF to AH; let the curves DKE, HKF, intersect each other in the point K; and join KB.

Because DA is to AH as AS to SB, KB will [cor. prop. 25.] be perpendicular to AS; because SA is to AB as SC to CB, the space DACE will [27.] be equal to the space HACF; and because DA is to AH as AS to SB, that is, as the centripetal to the centrifugal force at the point A, it is evident, that if from any point L, in AC, there be drawn LM perpendicular to AS, meeting the curves DKE, HKF, in M, N, the

centripetal force at the point L will be to the centrifugal at the point L as ML to LN; and because at the point A the centripetal force is greater than the centrifugal, therefore the body will descend from A towards the centre S. Suppose in any time it descends from A to any point L between AC, the sum of the centripetal forces that have acted upon the body in its descent from A to L will be to the sum of the centrifugal forces that have acted upon the body in its descent from A to L as the space DALM to the space HALN: but the space DALM is [cor. prop. 27.] greater than the space HALN; therefore the sum of the centripetal forces that have acted upon the body in its descent from A to L is greater than the sum of the centrifugal forces that have acted upon the body in its descent from A to L; therefore the body will descend from L towards S. Let it descend from L to any point O, between L, C, the same way it may be shown, that it will descend below the point O, and so on, till it arrive at the point C: and because the sum of the centripetal forces that have acted

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ed upon the body in its descent from A to C is to the sum of the centrifugal forces that have acted upon the body in its descent from A to C as the space DACE to the space HACF, and the space DACE is [27.] equal to the space HACF; therefore the sum of the centripetal forces that have acted upon the body in its descent from A to C will be equal to the sum of the centrifugal forces that have acted upon the body in its descent from A to C; therefore at the point C the sum of the centripetal and centrifugal forces balance each other; therefore the body will not descend below the point C.

P R O P. XXIX. *Fig. 54.*

Let there be a straight line CS; and suppose the body at the point C to be acted upon by two forces, a centrifugal and centripetal force, with respect to the centre S, the centrifugal to be reciprocally as the cube of the distance from the centre, and the centripetal to be reciprocally as the square of the distance from the centre; at the

at the point C let the centrifugal be greater than the centripetal; in SC produced take SB so, that CS may be to SB as the centripetal to the centrifugal force at the point C; in SB produced take SA so, that SA may be to AB as SC to CB: a body acted upon by the above forces will ascend from C to A, A and no higher.

From C draw CF perpendicular to SC, and in CF take CE, CF, so, that EC may be to CF as CS to SB; and let DKE, HKF, be two curves so related to AS, that if from any two points A, C, in AS, there be drawn AD, CE, perpendicular to AS, meeting the curves DKE, HKF, in D, E, and H, F, the square of AS is to the square of SC as CE to AD, likewise the cube of AS is to the cube of SC as CF to AH; let the curves DKE, HKF, intersect each other in the point K; and join KB.

Because EC is to CF as CS to SB, KB will [cor. prop. 25.] be perpendicular to AS; and because SA is to AB as SC to CB, the space DACE will [27.] be equal to the space HACF; and because EC is to CF as

CS

CS to SB, that is, as the centripetal to the centrifugal force at the point C, it is evident, that if from any point O, in CA, there be drawn OP, perpendicular to SC, meeting the curves DKE, HKF, in P, Q, the centripetal force at the point O will be to the centrifugal at the point O as PO to OQ; and because at the point C the centrifugal force is greater than the centripetal, therefore the body will ascend from C towards A. Suppose in any time it ascends from C to any point O, in AC, the sum of the centrifugal forces that have acted upon the body in its ascent from C to O will be to the sum of the centripetal forces that have acted upon the body in its ascent from C to O as the space FCOQ to the space ECOP: but the space FCOQ is [cor. prop. 27.] greater than the space ECOP; therefore the sum of the centrifugal forces that have acted upon the body in its ascent from C to O is greater than the sum of the centripetal forces that have acted upon the body in its ascent from C to O; therefore the body will ascend from O towards A. Let it ascend from O to any point L in OA, the
same

same way it may be shown, that it will ascend above the point L, and so on, till it arrive at the point A : and because the sum of the centrifugal forces that have acted upon the body in its ascent from C to A is to the sum of the centripetal forces that have acted upon the body in its ascent from C to A as the space HACF to the space DACE ; and the space HACF is [27.] equal to the space DACE ; therefore the sum of the centrifugal forces that have acted upon the body in its ascent from C to A will be equal to the sum of the centripetal forces that have acted upon the body in its ascent from C to A ; therefore, at the point A, the sum of the centrifugal and centripetal forces balance each other ; therefore the body will not ascend above the point A. Q. E. D.

P R O P. XXX. *Fig. 55.*

Let A be a given point in the straight line AB, and let CD, EF, be two curves on the same side AB, so related to the line AB, that if from any two points G, H, in AB, there be drawn GC, HD, perpendicular

to AB, meeting the curve CD in C, D, and the curve EF in E, F, the square of GA is to the square of AH as HD to GC, and likewise the cube of GA is to the cube of AH as HF to GE; let K be the point where the two curves intersect each other; draw KB perpendicular to AB, meeting AB in B; in AB take any two points G, L, so, that AG may be to GB as AL to LB; let the rectangle LGO be equal to the rectangle AGC: if from any point H in AB, there be drawn HD perpendicular to AB, meeting the curves CD, EF, in D, F, the square of AH will be to the rectangle GHL as the rectangle BGO to the excess of the space GHDC above the space EGHF.

Produce GA to P; and let GA, AP, be equal; from L draw LQ in any angle to AL; and let LQ be equal to AL; join AQ, PQ; from H draw HR parallel to LQ, meeting AQ, PQ, in R, S; and draw ST parallel to AH, meeting AQ in T.

Because [11.] the rectangle AGC is to the
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space

space CGHD as AH to HG, that is, as the rectangle AHP to the rectangle PHG, in reverse, the space CGHD will be to the rectangle AGC as the rectangle PHG to the rectangle PHA: because PA or GA is to AL as GB to BL; by composition, PL will be to LA as GL to LB; therefore PL will be to LA as PG to AB; and therefore PL will be to twice LA as PG to twice AB; that is, as GA to AB: but because the rectangle AGC is to the rectangle AGE as CG to GE, that is, [25.] as GA to AB; therefore the rectangle AGC is to the rectangle AGE as PL to twice LA: but PL is to twice LA or twice LQ as PH to twice HS; that is, as the rectangle PHA to twice the rectangle SHA; therefore the rectangle AGC is to the rectangle AGE as the rectangle PHA to twice the rectangle SHA: but because the space CGHD is to the rectangle AGC as the rectangle PHG to the rectangle PHA, and the rectangle AGC is to the rectangle AGE as the rectangle PHA to twice the rectangle SHA; therefore, by equality, the space CGHD will be to the rectangle AGE as the rectangle PHG to twice the rectangle SHA: but the rectangle

rectangle AGE is to the space EGHF [18.Tr. 1.] as twice the square of AH to the rectangle PHG; therefore, by perturbate equality, the space CGHD will be to the space EGHF as twice the square of AH to twice the rectangle SHA; that is, as the square of AH to the rectangle SHA: but the square of AH is to the rectangle SHA as AH to HS; therefore the space CGHD is to the space EGHF as AH to HS.

Again, because the space CGHD is to the space EGHF as AH to HS; that is, as RH to HS; conversely, the space CGHD will be to the excess of the space CGHD above the space EGHF as HR to RS; that is, as AH to ST; that is, as the square of AH to the rectangle contained by AH, ST; therefore the square of AH will be to the rectangle contained by AH, ST, as the space CGHD to the excess of the space CGHD above the space EGHF: but the rectangle contained by AH, ST, is to the rectangle contained by GH, ST, as AH to HG; that is, [11.] as the rectangle AGC to the space CGHD; therefore, by perturbate equality, the square of AH will

be to the rectangle contained by GH, ST, as the rectangle AGC to the excess of the space CGHD above the space EGHF. Again, because PA or AG is to AL as GB to BL, AP will be to PL as BG to GL; and because AP is to ST as PQ to QS, that is, as PL to LH; alternating, AP will be to PL as ST to LH; therefore ST will be to LH as GB to GL; therefore the rectangle contained by ST, GH, will be to the rectangle GHL as BG to GL; that is, as the rectangle BGO to the rectangle LGO; that is, as the rectangle BGO to the rectangle AGC: but because the square of AH is to the rectangle contained by GH, ST, as the rectangle AGC to the excess of the space CGHD above the space EGHF; and the rectangle contained by GH, ST, is to the rectangle GHL as the rectangle BGO to the rectangle AGC; therefore, by perturbate equality, the square of AH will be to the rectangle GHL as the rectangle BGO to the excess of the space CGHD above the space EGHF. Q. E. D.

PROP.

PROP. XXXI. Fig. 56.

Supposing a body to be acted upon by two forces, a centripetal and centrifugal force, with respect to the centre S, and the centripetal force to be reciprocally as the square of the distance from the centre, likewise the centrifugal force to be reciprocally as the cube of the distance from the centre; and at the point A let the centripetal force be greater than the centrifugal; join AS; in AS take the point B; and let AS be to SB as the centripetal force at the point A to the centrifugal force at the point A; in SB take the point C, and let SC be to CB as SA to AB; upon AC let there be a semicircle described; from any two points D, E, in AC, let there be drawn DF, EG, perpendicular to AC, meeting the semicircle in F, G; and join SG, meeting DF in H: the velocity at D will be to the velocity at E as FD to DH.

Draw AK perpendicular to AS; and in AK take AL so, that AK may be to AL

as the centripetal force at the point A to the centrifugal force at the point A; that is, as AS to SB; let KMN, LOP, be two curves, such, that if from any two points A, E, in AS, there be drawn AK, EM, perpendicular to AS, meeting the curve KMN in K, M, and likewise meeting the curve LOP in L, O, the square of AS is to the square of SE as EM to AK, and likewise the cube of AS is to the cube of SE as EO to AL; let GE, FD, meet the curve KMN in M, N, and likewise meet the curve LOP in O, P; and let the rectangle CAQ be equal to the rectangle SAK.

Because the square of SD is to the rectangle ADC, that is, to the square of FD [30.], as the rectangle BAQ to the excess of the space KADN above the space LADP; inversely, the square of FD will be to the square of DS as the excess of the space KADN above the space LADP to the rectangle BAQ: but the square of SD is to the square of DH as the square of SE to the square of EG, or the rectangle AEC; that is, [30.] as the rectangle BAQ to the excess of the space KAEM above the space LAEO;

LAEO; therefore, by equality, the square of FD is to the square of DH as the excess of the space KADN above the space LADP to the excess of the space KAEM above the space LAEO; therefore FD is to DH in the subduplicate ratio of the excess of the space KADN above the space LADP to the excess of the space KAEM above the space LAEO: but the velocity at D is to the velocity at E in the subduplicate ratio of the excess of the space KADN above the space LADP to the excess of the space KAEM above the space LAEO; therefore the velocity at D is to the velocity at E as FD to DH. *Q. E. D.*

P R O P. XXXII. Fig. 56.

Supposing a body to be acted upon by two forces, a centripetal and centrifugal force, with respect to the centre S; and at the point C let the centrifugal force be greater than the centripetal; in SC produced take SB so, that SB may be to SC as the centrifugal force at the point C to the centripetal force at the point C; in SB produced take
the

to the point A; and let SA be to AB as SC to CB; upon AC let there be a semicircle described; from any two points D, E, in AC, let there be drawn DF, EG, perpendicular to AC, meeting the semicircle in F, G; and join SG, meeting DF in H: the velocity at D will be to the velocity at E as FD to DH.

Draw CT perpendicular to CS, and in CT take CR so, that CT may be to CR as the centrifugal force at the point C to the centripetal force at the point C; that is, as SB to SC; let RNM, TPO, be two curves, such, that if from any two points C, E, in AS, there be drawn CR, EM, perpendicular to AS, meeting the curve RNM in R, M, and likewise meeting the curve TPO in T, O, the square of CS is to the square of SE as EM to CR, and likewise the cube of CS is to the cube of SE as EO to CT; let GE, FD, meet the curve RNM in M, N, and likewise meet the curve TPO in O, P; draw AK perpendicular to SA, meeting the curves RNM,

RNM, TPO, in K, L; and let the rectangle CAQ be equal to the rectangle SAK.

Because the square of SD is to the rectangle ADC, that is, to the square of FD [30.] as the rectangle BAQ to the excess of the space KADN above the space LADP; inversely, the square of FD will be to the square of DS as the excess of the space KADN above the space LADP to the rectangle BAQ: but the square of SD is to the square of DH as the square of SE to the square of EG or the rectangle AEC; that is, [30.] as the rectangle BAQ to the excess of the space KAEM above the space LAEO; therefore, by equality, the square of FD is to the square of DH as the excess of the space KADN above the space LADP to the excess of the space KAEM above the space LAEO: but because the space KACR is [27.] equal to the space LACT, the excess of the space KADN above the space LADP will be equal to the excess of the space TCDP above the space RCDN; and the excess of the space KAEM above the space LAEO will be equal to the excess of the space TCEO above the space RCEM;

X

therefore

therefore the square of FD is to the square of DH as the excess of the space TCDP above the space RCDN to the excess of the space TCEO above the space RCEM; therefore FD is to DH in the subduplicate ratio of the excess of the space TCDP above the space RCDN to the excess of the space TCEO above the space RCEM: but the velocity at D is to the velocity at E in the subduplicate ratio of the excess of the space TCDP above the space RCDN to the excess of the space TCEO above the space RCEM; therefore the velocity at D is to the velocity at E as FD to DH. $\mathcal{Q}$. E. D.

P R O P. XXXIII. *Fig. 57.*

Let there be a semicircle whose diameter is AB, and centre C; let D, E, be two points in AC, such, that the rectangle DCE is equal to the square of AC; and let the point D be in BA produced; let FGH be a curve, such, that if from any point G in the curve there be drawn GK perpendicular to AB, meeting AB in K, and the semicircle in L, the rectangle GKL is equal

equal to the rectangle contained by DK, CE; it is evident that AM, BN, drawn perpendicular to AB, are asymptotes to the curve FGH. If from any point L in the semicircle there be drawn LK perpendicular to AB, meeting AB in K, and the curve FGH in G, and EL be joined; the space contained by GK, KB, the asymptote BN, and the curve GH indefinitely produced, is double of the sector BEL; likewise the space contained by AB, and the asymptotes AM, BN, and the curve FGH indefinitely produced, is double of the semicircle ALB.

Suppose the arc LB to be divided into indefinitely little parts in $a, b, c, d, \&c.$; from $a, b, c, d, \&c.$ let there be drawn perpendiculars to AB, meeting AB in $e, f, g, h, \&c.$ and the curve FGH in $k, l, m, n, \&c.$; join Ea, Eb, Ec, Ed, &c.; let LO be a tangent to the semicircle at L; and draw EO perpendicular to LO, meeting LO in O; join CL; and draw ao parallel to CB, meeting KL, LC, in $o, p.$

Because the rectangle DCE is equal to the square of CA; therefore [4.] EO will

X 2

will be to DK as EC to CA or CL; therefore the rectangle contained by EO, CL, will be equal to the rectangle contained by DK, CE: but, from the nature of the curve FGH, the rectangle GKL is equal to the rectangle contained by DK, CE; therefore the rectangle contained by EO, CL, is equal to the rectangle GKL; and therefore GK is to EO as CL to LK; that is, as pL to Lo : but because the triangles pLo , aoL , are similar, pL is to Lo as aL to ao or Ke ; therefore GK is to EO as aL to Ke ; therefore the rectangle Ge is equal to the rectangle contained by EO, aL : but the rectangle contained by EO, aL , is double of the triangle aEL ; therefore the rectangle Ge is double of the triangle aEL . The same way it is shown, that the rectangle kf is double of the triangle aEb ; and the rectangle lg double of the triangle bEc , and the rectangle mb double of the triangle cEd , and so on; therefore the sum of the rectangles Ge , kf , lg , mb , &c. is double of the sum of the triangles aEL , aEb , bEc , cEd , &c.: but the sum of the rectangles Ge , kf , lg , mb , &c. make up the space contained

contained by GK, KB, the asymptote BN, and the curve GH indefinitely produced, and the sum of the triangles $\triangle EL$, $\triangle Eb$, $\triangle Ec$, $\triangle Ed$, &c. make up the sector BEL; therefore the space contained by GK, KB, the asymptote BN, and the curve GH indefinitely produced, is double of the sector BEL. From this it is evident likewise, that the space contained by AB, and the asymptotes AM, BN, and the curve FGH indefinitely produced, is double of the semicircle AEB. Q. E. D.

P R O P. XXXIV. Fig. 58.

Let there be a semicircle whose diameter is AC, and centre H; let S, B, be two points in HC, such, that the rectangle SHB is equal to the square of CH; and let the point S be without the semicircle; suppose a body to descend in AC, such, that if from any two points D, E, in AC, there be drawn DF, EG, perpendicular to AC, meeting the semicircle in F, G, and SF be joined meeting EG in Q, the velocity of the body at E is to the velocity of the body at D as GE

to EQ; join BF, BG: the time of descent from A to D will be to the time of descent from A to E as the sector ABF to the sector ABG; likewise the time of descent from A to D will be to the time of descent from A to C as the sector ABF to the semicircle AFC.

Let there be a curve KLMN, so related to the semicircle, that if from any point F in the semicircle there be drawn FD perpendicular to AC, meeting AC in D, and the curve KLMN in L, the rectangle FDL is equal to the rectangle contained by SD, BH; let EG meet the curve KLMN in M; join SF; and let SF meet EG in Q; draw AO, CP, perpendicular to AC; it is evident, that AO, CP, are asymptotes to the curve KLMN.

Because from the nature of the curve KLMN, the rectangle FDL is equal to the rectangle contained by SD, BH, and the rectangle GEM equal to the rectangle contained by SE, BH; therefore the rectangle FDL will be to the rectangle GEM as the rectangle contained by SD, BH, to the rectangle

rectangle contained by SE, BH; that is, as SD to SE: but SD is to SE as FD to QE; that is, as the rectangle FDL to the rectangle contained by QE, DL; therefore the rectangle FDL is to the rectangle GEM as the rectangle FDL to the rectangle contained by QE, DL; therefore the rectangle GEM is equal to the rectangle contained by QE, DL; therefore LD is to EM as EG to EQ: but the velocity at E is to the velocity at D as EG to EQ; therefore LD is to EM as the velocity at E is to the velocity at D; and therefore [4. Tr. 1.] the time of descent from A to D will be to the time of descent from A to E as the space contained by LD, DA, the asymptote AO, and the curve MLK indefinitely produced, to the space contained by ME, EA, the asymptote AO, and the curve MLK indefinitely produced: but [33.] the space contained by LD, DA, the asymptote AO, and the curve MLK indefinitely produced, is double of the sector ABF; and the space contained by ME, EA, the asymptote AO, and the curve MLK indefinitely produced, is double of the

the sector ABG; therefore the space contained by LD, DA, the asymptote AO, and the curve MLK indefinitely produced, will be to the space contained by ME, EA, the asymptote AO, and the curve MLK indefinitely produced, as the sector ABF to the sector ABG; therefore the time of descent from A to D will be to the time of descent from A to E as the sector ABF to the sector ABG. The same way it is shown, that the time of descent from A to D is to the time of descent from A to C as the sector ABF to the semicircle AFC.
 $\mathcal{Q}. E. D.$

COR. I. The same construction remaining as in the proposition, if a body ascend from C towards A, and the velocity of the body at E be to the velocity of the body at D as GE to EQ, the time of ascent from C to E will be to the time of ascent from C to D as the sector CBG to the sector CBF; likewise the time of ascent from C to E will be to the time of ascent from C to A as the sector CBG to the semicircle AGC.

It

It was shown, that LD is to EM as GE to EQ, and the velocity at E is to the velocity at D as GE to EQ; therefore the velocity at E is to the velocity at D as LD to EM; therefore the time of ascent from C to E will be to the time of ascent from C to D as the space contained by CE, EM, the curve MN, and its asymptote CP indefinitely produced, to the space contained by CD, DL, the curve LN, and its asymptote CP indefinitely produced: but [33.] the space contained by CE, EM, the curve MN, and its asymptote CP indefinitely produced, is double of the sector CBG; and the space contained by CD, DL, the curve LN, and its asymptote CP indefinitely produced, is double of the sector CBF; therefore the time of ascent from C to E will be to the time of ascent from C to D as the sector CBG to the sector CBF. The same way it is shown, that the time of ascent from C to E is to the time of ascent from C to A as the sector CBG to the semicircle AGC.

COR. 2. Fig. 59. Supposing a body to

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be

be acted upon by two forces, a centripetal and centrifugal force, with respect to the centre S , and the centripetal force to be reciprocally as the square of the distance from the centre, likewise the centrifugal force to be reciprocally as the cube of the distance from the centre; and at the point A let the centripetal force be greater than the centrifugal; join AS ; in AS take the point B , and let AS be to SB as the centripetal force at the point A to the centrifugal force at the point A ; in SB take the point C , and let SC be to CB as SA to AB ; upon AC let there be a semicircle described; from any two points D, E , in AC , let there be drawn DF, EG , perpendicular to AC , meeting the semicircle in F, G ; join BF, BG : the time of descent from A to D will be to the time of descent from A to E as the sector ABF to the sector ABG ; likewise the time of descent from A to D will be to the time of descent from A to C as the sector ABF to the semicircle AFC .

Join SG , meeting DF in H : because the velocity at D is [31.] to the velocity at E as
FD

FD to DH; therefore [34.] the time of descent from A to D will be to the time of descent from A to E as the sector ABF to the sector ABG; likewise it is evident, that the time of descent from A to D will be to the time of descent from A to C as the sector ABF to the semicircle AFC.

COR. 3. Supposing a body to be acted upon by two forces, a centrifugal and centripetal force, with respect to the centre S, and the centrifugal force to be reciprocally as the cube of the distance from the centre, likewise the centripetal force to be reciprocally as the square of the distance from the centre; and at the point C let the centrifugal force be greater than the centripetal; in SB produced take the point A, and let SA be to AB as SC to CB; upon AC let there be a semicircle described; from any two points D, E, in AC, let there be drawn DF, EG, perpendicular to AC, meeting the semicircle in F, G; join BF, BG: the time of ascent from C to D will be to the time of ascent from C to E as the sector CBF to the sector CBG; likewise

the time of ascent from C to D will be to the time of ascent from C to A as the sector CBF to the semicircle AFC.

Because the velocity at D is [32.] to the velocity at E as FD to DH; therefore [cor. 1. prop. 34.] the time of ascent from A to D will be to the time of ascent from A to E as the sector CBF to the sector CBG; likewise it is evident, that the time of ascent from C to D will be to the time of ascent from C to A as the sector CBF to the semicircle AFC.

P R O P. XXXV. Fig. 60.

In the straight line AS let B be any given point; in SB take the point C so, that SC may be to CB as SA to AB; suppose a body at the point A to be acted upon by two forces, a centripetal and centrifugal force, with respect to the centre S; and let the centripetal force at the point A be to the centrifugal force at the point A as AS to SB: the body therefore will [28.] descend from A to C, and from the point C will ascend to A, and vibrate perpetually from A to C
and

and from C to A; likewise the time of descent from A to C will be equal to the time of ascent from C to A.

Because the centripetal force at the point C is to the centripetal force at the point A as the square of SA to the square of SC; that is, as the parallelopiped whose base is the square of SA, and altitude SB, to the parallelopiped whose base is the square of SC, and altitude SB; and the centripetal force at the point A is to the centrifugal force at the point A as AS to SB; that is, as the cube of AS to the parallelopiped whose base is the square of AS, and altitude SB; therefore, by perturbate equality, the centripetal force at C will be to the centrifugal force at A as the cube of AS to the parallelopiped whose base is the square of SC, and altitude SB: but the centrifugal force at A is to the centrifugal force at C as the cube of SC to the cube of SA; therefore, by perturbate equality, the centripetal force at C will be to the centrifugal force at C as the cube of SC to the parallelopiped whose base is the square of SC,

SC, and altitude SB; that is, as SC to SB; therefore, inversely, the centrifugal force at C will be to the centripetal force at C as SB to SC; and therefore [29.] the body will ascend from C to A: and therefore it is evident, that the body will vibrate perpetually from A to C and from C to A.

Again, the time of descent from A to C will be equal to the time of ascent from C to A.

Let AD be passed over by the body in an indefinitely little part of time when descending from A to C; likewise let CF be passed over by the body when ascending from C to A, in a time equal to that in which AD was passed over; draw DE, FG, perpendicular to AC, meeting the semicircle described upon AC, in E, G; and join BE, BG.

Because the centripetal force at the point A is to the centrifugal force at the point A as AS to SB; conversely, the centripetal force at A will be to the excess of the centripetal force at A above the centrifugal force at A as SA to AB; that is, as SC to CB; that is, as the square of SC to the
rectangle

rectangle SCB; therefore, inversely, the excess of the centripetal force at A above the centrifugal force at A will be to the centripetal force at A as the rectangle SCB to the square of SC: but the centripetal force at A is to the centripetal force at C as the square of SC to the square of SA; therefore, by equality, the excess of the centripetal force at A above the centrifugal force at A will be to the centripetal force at C as the rectangle SCB to the square of SA. Again, because the centrifugal force at C is to the centripetal force at C as SB to SC, by division, the excess of the centrifugal force at C above the centripetal force at C will be to the centripetal force at C as BC to CS; that is, as the rectangle SCB to the square of CS; therefore, inversely, the centripetal force at C will be to the excess of the centrifugal force at C above the centripetal force at C as the square of SC to the rectangle SCB: but because the excess of the centripetal force at A above the centrifugal force at A is to the centripetal force at C as the rectangle SCB to the square of AS, and the centripetal force at C is to the

the excess of the centrifugal force at C above the centripetal force at C as the square of SC to the rectangle SCB; therefore, by perturbate equality, the excess of the centripetal force at A above the centrifugal force at A will be to the excess of the centrifugal force at C above the centripetal force at C as the square of SC to the square of SA: but, because AD, CF, are passed over in equal times, and the time is supposed to be indefinitely little, AD will be to CF as the excess of the centripetal force at A above the centrifugal force at A to the excess of the centrifugal force at C above the centripetal force at C; therefore AD is to CF as the square of SC to the square of SA: but the rectangle CAD is to the rectangle ACF as AD to CF; therefore the rectangle CAD is to the rectangle ACF as the square of SC to the square of SA: but, because AD, CF, are indefinitely little, the rectangles CAD, CDA, may be considered as equal; likewise the rectangles ACF, AFC, may be considered as equal; therefore the rectangle ADC is to the rectangle AFC as the square of SC to the square of

of

Plate 8

Fig 53

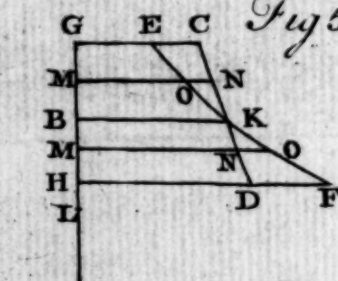


Fig 54

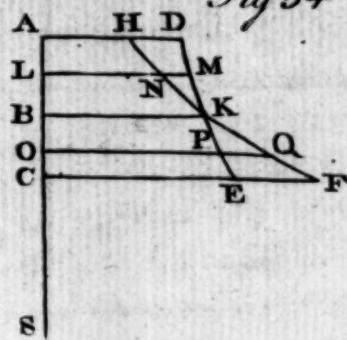


Fig 55

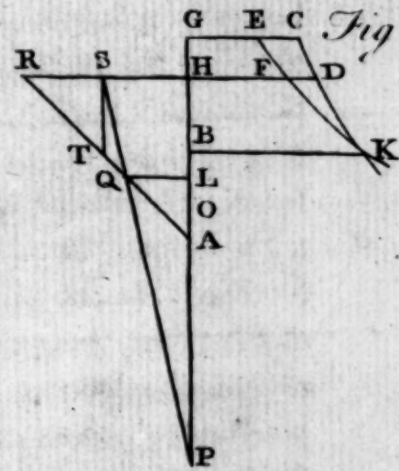


Fig 56

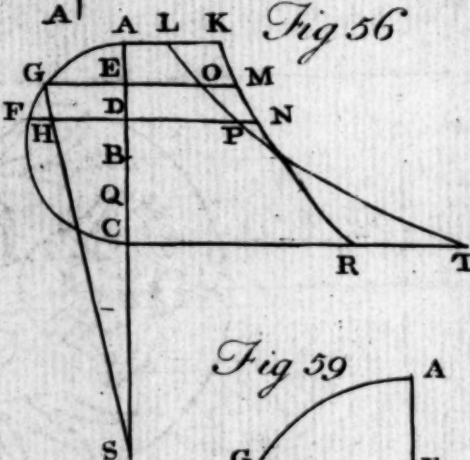


Fig 57

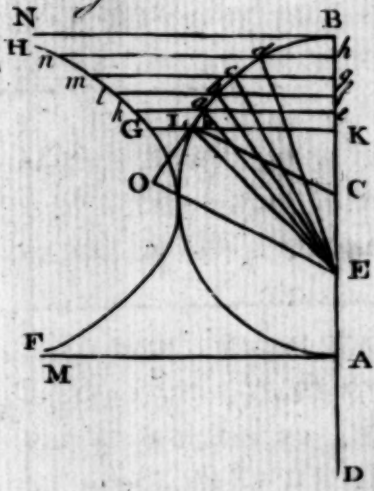


Fig 59

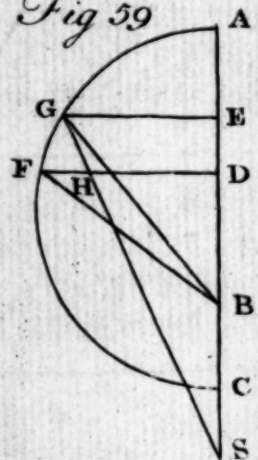
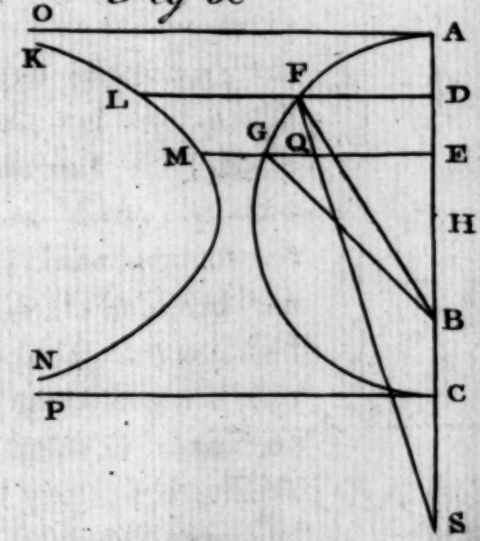


Fig 58



of SA: but, because of the semicircle, the rectangle ADC is equal to the square of DE, and the rectangle AFC equal to the square of FG; therefore the square of DE is to the square of FG as the square of SC to the square of SA; therefore DE is to FG as SC to SA: but because SC is to CB as SA to AB, alternating, SC is to SA as CB to BA; therefore ED is to FG as CB to BA; therefore the rectangle contained by BA, ED, is equal to the rectangle contained by BC, GF; but the rectangle contained by BA, ED, is double of the triangle ABE; and the rectangle contained by CB, FG, is double of the triangle CBG; therefore the triangle ABE is equal to the triangle CBG; that is, the sector ABE is equal to the sector CBG. Because the time of descent from A to C is to the time of descent from A to D [cor. 2. prop. 34.] as the semicircle AEC to the sector ABE, and the time of descent from A to D is equal to the time of ascent from C to F; therefore the time of descent from A to C is to the time of ascent from C to F as the semicircle AEC to the sector ABE:

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but

but the time of ascent from C to F is to the time of ascent from C to A [cor. 3. prop. 34.] as the sector CBG to the semicircle AEC; therefore, by perturbate equality, the time of descent from A to C is to the time of ascent from C to A as the sector CBG to the sector ABC: but the sector CBG is equal to the sector ABE; therefore the time of descent from A to C is equal to the time of ascent from C to A. Q. E. D.

P R O P. XXXVI. Fig. 61.

Supposing a body acted upon by two forces, a centripetal and centrifugal, with respect to the centre S, the centripetal force tending to the centre S to be reciprocally as the square of the distance from the centre, and the centrifugal force to be reciprocally as the cube of the distance from the centre; let the centripetal force at the point A be greater than the centrifugal force at the same point A; join AS; in AS take the point B; and let AS be to SB as the centripetal force at the point A to the centrifugal force at the same point A; and in

AB

AB take the point C, such, that SC is to CB as SA to AB: it is required to find the height SD, from which a body acted upon by a centripetal force that is reciprocally as the square of the distance from the centre would fall to the centre S, in the same time that a body acted upon by the centripetal and centrifugal forces above mentioned would fall from the point A to the point C,

Let the square of SD be to the square of of SA as SA to AK, and let the rectangle BAL be equal to the square of AC; let AE be described in an indefinitely little part of time by the body falling from A towards C; and let DF be described by the body falling from D to S in the same time that the part AE was described; upon AC, DS, let the semicircles AGC, DHS, be described; draw EG perpendicular to AC, meeting the semicircle AGC in G; draw FH perpendicular to SD, meeting the semicircle DHS in H; join BG, SH.

Because the centripetal force at A is to the centrifugal force at the same point A as AS to SB, the centripetal force at A will

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be

be to the excess of the centripetal above the centrifugal force at the same point A as AS to AB; therefore the excess of the centripetal above the centrifugal force at the point A will be to the centripetal force at the same point A as AB to AS; but the centripetal force at A is to the centripetal force at D as the square of DS to the square of SA; that is, as SA to AK; therefore the excess of the centripetal above the centrifugal force at the point A is to the centripetal force at the point D as BA to AK: but, because AE is described in the same time that DF is described, AE will be to DF as the excess of the centripetal and centrifugal forces at the point A to the centripetal force at the point D; therefore AE is to DF as BA to AK. Again, because the time of descent from A to E is to the time of descent from A to C as [cor. 2. prop. 34.] the sector ABG to the semicircle AGC, and the time of descent from D to F is to the time of descent from D to S as [15.] the sector DSH to the semicircle DHS, and the time of descent from A to E is equal to the time of descent from D to F, likewise the time of descent from A to C is equal to the time of descent from

from D to S; therefore the sector ABG will be to the semicircle AGC as the sector DSH to the semicircle DHS; therefore, alternating, the sector ABG will be to the sector DSH as the semicircle AGC to the semicircle DHS: but the sector ABG is to the sector DSH as the rectangle contained by AB, GE, to the rectangle contained by DS, HF; and the semicircle AGC is to the semicircle DHS as the square of AC to the square of DS; that is, as the rectangle BAL to the square of DS; therefore the rectangle contained by BA, GE, is to the rectangle contained by DS, HF, as the rectangle BAL to the square of DS; therefore, alternating, the rectangle contained by BA, GE, is to the rectangle BAL as the rectangle contained by SD, HF, to the square of SD: but the rectangle contained by BA, GE, is to the rectangle BAL as GE to AL, and the rectangle contained by SD, HF, is to the square of SD as HF to SD; therefore GE is to AL as HF to SD; therefore the square of GE is to the square of AL as the square of HF to the square of SD; that is, the rectangle AEC is to the square of AL as the rectangle DFS

to

to the square of SD : but, because AE , DF , are indefinitely little, the rectangles AEC , CAE , may be considered as equal; likewise the rectangles DFS , SDF , may be considered as equal; therefore the rectangle CAE will be to the square of AL as the rectangle SDF to the square of SD ; that is, as DF to SD ; that is, as the rectangle contained by CA , DF , to the rectangle contained by CA , SD ; and therefore the rectangle CAE will be to the rectangle contained by CA , DF , as the square of AL to the rectangle contained by CA , SD ; therefore AE will be to DF as the square of AL to the rectangle contained by CA , SD : but AE is to DF as BA to AK ; therefore the square of AL is to the rectangle contained by CA , SD , as BA to AK .

Again, because the rectangle BAL is equal to the square of AC , AL will be to AC as AC to AB ; therefore the square of AL will be to the square of AC as the square of AC to the square of AB ; that is, as the parallelopiped whose base is the square of AC , and altitude SD , to the parallelopiped whose base is the square of AB , and altitude

tude SD : but the square of AC is to the rectangle contained by AC , SD , as AC to SD ; that is, as the cube of AC to the parallelopiped whose base is the square of AC , and altitude SD ; therefore the square of AL is to the rectangle contained by AC , SD , as the cube of AC to the parallelopiped whose base is the square of AB , and altitude SD . Again, because BA is to AS as the parallelopiped whose base is the square of SD , and altitude BA , to the parallelopiped whose base is the square of SD , and altitude SA ; and SA is to AK as the square of SD to the square of SA ; that is, as the parallelopiped whose base is the square of SD , and altitude SA , to the cube of SA ; therefore, by equality, BA will be to AK as the parallelopiped whose base is the square of SD , and altitude BA , to the cube of AS : but because the square of AL is to the rectangle contained by CA , SD , as BA to AK , and the square of AL is to the rectangle contained by CA , SD , as the cube of AC to the parallelopiped whose base is the square of AB , and altitude SD , and BA is to AK as the parallelopiped whose base is the square of SD , and altitude BA ,
 BA ,

BA, to the cube of AS; therefore the cube of AC will be to the parallelopiped whose base is the square of AB, and altitude SD, as the parallelopiped whose base is the square of SD, and altitude BA, to the cube of SA; and therefore the cube of AC will be to the parallelopiped whose base is the square of SD, and altitude BA, as the parallelopiped whose base is the square of BA, and altitude SD, to the cube of SA: but the parallelopiped whose base is the square of SD, and altitude BA, is to the cube of SD as BA to SD; that is, as the cube of BA to the parallelopiped whose base is the square of BA, and altitude SD; therefore the cube of AC is to the cube of SD as the cube of BA to the cube of AS; and therefore AC is to SD as BA to AS; therefore CA is to AB as DS to SA: but, because SC is to CB as SA to AB, alternating, SC will be to SA as CB to BA; therefore, by composition, CS, together with SA, will be to SA as CA to AB; therefore DS is to SA as CS, together with SA, is to SA; and therefore DS is equal to CS, together with SA; therefore AD is equal to CS.

Q. E. I.

TRACT

TRACT III.

PROP. I. Fig. 62.

LET there be a semicircle whose diameter is AB; from any point C in AB, draw CD perpendicular to AB, meeting the semicircle in D; in CD take CE equal to BC; and join AE; from any point F in AB draw FG perpendicular to AB, meeting AE in G, and the semicircle in H; and join BH, meeting CD in K: the square of DC will be to the square of CK as BF to FG.

Because the square of DC is equal to the rectangle ACB, and BC, CE, are equal, the square of DC will be to the square of CE as the rectangle ACB to the square of CB; that is, as AC to CB, or CE; that is, as AF to FG. Again, because BC, CE, are equal, the square of CE will be to the square of CK as the square of BC to the square of CK; that is, as the square of

A a

BF

BF to the square of FH; that is, as the square of BF to the rectangle BFA; that is, as BF to FA: but because the square of DC is to the square of CE as AF to FG, and the square of GE is to the square of CK as BF to FA; therefore, by perturbate equality, the square of DC will be to the square of CK as BF to FG. *Q. E. D.*

P R O P. II. *Fig. 63.*

Suppose a body acted upon by a centripetal force tending to the centre C, to descend in the straight line AC, and the centripetal force to be reciprocally as the square of the distance from the centre; in AC take any point B; draw BD equal to BC in any given angle; and join AD; from any point E in AC draw EF parallel to BD, meeting AD in F: supposing the body to fall from the point A, the velocity at B will be to the velocity at E in the subduplicate ratio of CE to EF.

Upon AC let there be a semicircle described; draw BG, EH, perpendicular to AC,

AC, meeting the semicircle in G, H; in BG take BK equal to BD or BC; join AK, meeting EH in L; and join CH, meeting BG in M.

Because BD is to EF as AB to AE; that is, as BK to EL; and BD, BK, are equal; therefore EF, EL, are equal; therefore CE is to EF as CE to EL; but [1.] the square of GB is to the square of BM as CE to EL; therefore the square of GB is to the square of BM as CE to EF; therefore GB is to BM in the subduplicate ratio of CE to EF: but the velocity at B is [12. Tr. 2.] to the velocity at E as GB to BM; therefore the velocity at B is to the velocity at E in the subduplicate ratio of CE to EF. Q. E. D.

P R O P. III. Fig. 64.

Supposing the law of centripetal force tending to a given centre to be reciprocally as the square of the distance from the centre; it is required to find the height from which a body must fall, in order to acquire a velocity equal to the velocity with which it de-

A a 2

scribes

scribes a circle at a given distance from the centre.

Let BEG be a circle whose centre is C; and suppose a body acted upon by a centripetal force tending to the centre C to revolve in the circle, and the centripetal force to be reciprocally as the square of the distance from the centre C; it is required to find in CB produced the point A so, that the body falling from A to B would acquire at the point B a velocity equal to its velocity in the circle.

Draw BF in any angle to AB; and let BF be equal to CB; and join AF; let AC meet the circle in G; let BD be an arc of the circle described in an indefinitely little part of time; draw DH perpendicular to BC, meeting BC in H; suppose the body to fall from A to K in a time equal to that in which the arc BD was described; draw KL parallel to BF, meeting AF in L; and let AC be to CB as CB to CM; let BN be a tangent to the circle in the point B; join CL, meeting BF in O; and draw DP parallel to CB, meeting BN in P.

Because

Because the arc BD is described in an indefinitely little part of time, BH will be indefinitely little; and therefore the rectangles GHB, GBH, may be considered as equal: but, because of the circle, the rectangle GHB is equal to the square of HD; therefore the rectangle GBH will be equal to the square of HD; that is, twice the rectangle CBH will be equal to the square of HD; therefore twice BH will be to HD as HD to BC; therefore twice BH will be to HD in the subduplicate ratio of twice BH to BC: but twice BH is to HD, or BP, as the velocity at H [10. Tr. 1.] to the velocity in BN; that is, as the velocity that would be acquired in falling from B to H to the velocity that would be acquired in falling from A to B; therefore the velocity that would be acquired in falling from B to H is to the velocity that would be acquired in falling from A to B in the subduplicate ratio of twice BH to BC; and the velocity that would be acquired in falling from A to B is to the velocity that would be acquired in falling from A to K [2.] in the subduplicate ratio of CK to KL; that is, in the subduplicate

subduplicate ratio of CB to BO; therefore the velocity that would be acquired in falling from B to H is to the velocity that would be acquired in falling from A to K in the subduplicate ratio of twice BH to BO: but because the times in which the body falls from A to K, and from B to H, are equal, and the time is supposed to be indefinitely little, the velocity at H will be to the velocity at K as the centripetal force at B to the centripetal force at A; that is, as the square of AC to the square of CB; that is, as AC to CM; therefore AC will be to CM in the subduplicate ratio of twice HB to BO; therefore the square of AC will be to the square of CM as twice HB to BO; and therefore the square of AC will be to twice the square of CM as twice HB to twice BO; that is, as HB to BO.

Again, because BH, AK, are described in equal times, and the time is supposed to be indefinitely little, BH will be to AK as the centripetal force at B to the centripetal force at A; that is, as the square of AC to the square of CB; that is, as the square of

CB

CB to the square of CM: but AK is to KL as AB. to BF or BC; that is, as the rectangle ABC to the square of BC; therefore BH will be to KL as the rectangle ABC to the square of CM: but because KL is to BO as KC to CH; that is, as AC to CB; that is, as the rectangle CAB to the rectangle ABC; therefore HB will be to BQ as the rectangle CAB to the square of CM: but it was shown, that the square of AC is to twice the square of CM as HB to BO; therefore the square of AC is to twice the square of CM as the rectangle CAB to the square of CM; therefore the square of AC is double of the rectangle CAB; and therefore CA is double of CB. *Q. E. I.*

PROP. IV. Fig. 65.

Let there be an ellipse whose greater axis is AB, and focus S; and suppose a body acted upon by a centripetal force tending to the focus S to revolve in the ellipse; it is required to find the height EA from which a body must fall to acquire at the point A a velocity equal to the velocity of the

the body at the point A revolving in the ellipse.

Let C be the centre of the ellipse, CD half the lesser axis; let the square of CD be equal to the rectangle contained by AC, SG, and the square of AS equal to the rectangle ESP; let ES be to SA as GS to SQ; from A draw AF in any angle to AS; let AF be equal to AS; and join EF; let AL be a tangent to the ellipse at A; and let AH be a part of the ellipse described in an indefinitely little part of time; draw HK perpendicular to AB, meeting AB in K; and complete the parallelogram AKHR; suppose the body to fall from E to M in a time equal to that in which AH was described; draw MN parallel to AF, meeting EF in N; join SN; and draw GO parallel to AF, meeting SN in O.

Because AH is described in an indefinitely little part of time, AK will be indefinitely little; and therefore the rectangles AKB, KAB, may be considered as equal; that is, twice the rectangle CAK may be considered as equal to the rectangle AKB: but

[Con.

[Con. 2. 6.] the rectangle AKB is to the square of HK as the square of AC to the square of CD; that is, as the square of AC to the rectangle contained by AC, SG; that is, as AC to SG; therefore twice the rectangle CAK will be to the square of HK as AC to SG; that is, as twice the rectangle CAK to twice the rectangle contained by SG, AK; therefore twice the rectangle contained by SG, AK, will be equal to the square of HK; therefore twice AK will be to KH as KH to SG; therefore twice AK will be to KH in the subduplicate ratio of twice AK to SG; but twice AK is to KH, or AR, [10. Tr. 1.] as the velocity at K to the velocity in AL; that is, as the velocity that would be acquired in falling from A to K to the velocity that would be acquired in falling from E to A; therefore the velocity that would be acquired in falling from A to K is to the velocity that would be acquired in falling from E to A in the subduplicate ratio of twice AK to SG: but because the body revolving in the ellipse is acted upon by a centripetal force tending to the focus S, the centripetal force will [24.

B b

Tr.

Tr. I.] be reciprocally as the square of the distance from the focus S; therefore the velocity that would be acquired in falling from E to A is to the velocity that would be acquired in falling from E to M [2.] in the subduplicate ratio of SM to MN; that is, in the subduplicate ratio of SG to GO; therefore the velocity that would be acquired in falling from A to K is to the velocity that would be acquired in falling from E to M in the subduplicate ratio of twice AK to GO: but because the times in which the body falls from A to K and from E to M are equal, and the time is supposed to be indefinitely little, the velocity at A will be to the velocity at E as the centripetal force at A to the centripetal force at E; that is, as the square of ES to the square of SA, or the rectangle ESP; that is, as ES to SP; therefore ES will be to SP in the subduplicate ratio of twice AK to GO; therefore the square of ES will be to the square of SP as twice AK to GO.

Again, because AK, EM, are described in equal times, and the time is supposed to be indefinitely little, AK will be to EM as the

the centripetal force at A to the centripetal force at E; that is, as the square of ES to the square of SA; that is, as ES to SP; that is, as the rectangle SEA to the rectangle contained by EA, SP: but EM is to MN as EA to AF, or AS; that is, as the rectangle contained by EA, SP, to the rectangle ASP; therefore AK is to MN as the rectangle SEA to the rectangle ASP: but MN is to GO as MS, or SE, to SG; that is, as AS to SQ; that is, as the rectangle ASP to the rectangle QSP; therefore AK is to GO as the rectangle SEA to the rectangle QSP; therefore twice AK is to GO as twice the rectangle SEA to the rectangle QSP: but the square of ES is to the square of SP as twice AK to GO; therefore the square of ES will be to the square of SP as twice the rectangle SEA to the rectangle QSP; therefore, alternating, the square of ES will be to twice the rectangle SEA as the square of SP to the rectangle QSP; therefore SE will be to twice EA as PS to SQ; but because ES is to SA as GS to SQ, inversely, AS will be to SE as QS to SG:

B b 2

and

and because SE is to twice EA as PS to SQ ; therefore, by perturbate equality, AS will be to twice AE as PS to SG ; therefore, alternating, AS will be to SP as twice AE to SG : but ES is to SA as SA to SP ; therefore SE is to SA as twice AE to SG ; therefore, alternating, ES will be to twice AE as AS to SG : but because the rectangle contained by CA , SG , is equal to the square of CD ; that is, equal to the rectangle ASB ; AS will be to SG as AC to SB ; therefore SE will be to twice AE as AC to SB ; that is, as AB to twice SB ; therefore SE will be to EA as AB to BS ; and therefore, by division, SA will be to AE as AS to SB ; therefore AE is equal to SB . Q. E. I.

PROP. V. Fig. 66.

Suppose a body acted upon by a centripetal force that is reciprocally as the square of the distance from the centre S , to describe the circle ABC ; join AS : it is required to find in AS the point E , so, that supposing a body to fall from A to E , it would acquire at the point

point E a velocity equal to the velocity of the body describing the circle.

Let AS meet the circle in C; let AB be an arc of the circle described in an indefinitely little part of time; draw BG perpendicular to AC, meeting AC in G; complete the parallelogram AGBD; draw EF in any angle to AE, let EF be equal to ES; join AF; and draw GH parallel to EF, meeting AF in H.

Because the arc AB is described in an indefinitely little part of time, AG will be indefinitely little; therefore the rectangles AGC, GAC, may be considered as equal: but because of the circle, the rectangle AGC is equal to the square of BG; therefore the rectangle GAC will be equal to the square of BG; that is, twice the rectangle SAG will be equal to the square of BG; therefore twice AG will be to BG as BG to AS; therefore twice AG will be to BG in the subduplicate ratio of twice AG to AS: but twice AG is [10. Tr. 1.] to BG, or AD, as the velocity at G to the velocity in AD; that is, as the velocity that would be acquired

acquired in falling from A to G to the velocity that would be acquired in falling from A to E; therefore the velocity that would be acquired in falling from A to G is to the velocity that would be acquired in falling from A to E in the subduplicate ratio of twice AG to AS: but the velocity that would be acquired in falling from A to G is to the velocity that would be acquired in falling from A to E [2.] in the subduplicate ratio of HG to GS; that is, (because GS, SA, may be considered as equal), in the subduplicate ratio of HG to AS; therefore the subduplicate ratio of twice AG to AS is equal to the subduplicate ratio of HG to AS; therefore twice AG is to AS as HG to AS; therefore HG is double of AG: but FE is to EA as HG to GA; therefore FE is double of EA: but FE is equal to SE; therefore SE is double of EA. Q. E. I.

P R O P. VI. *Fig. 67.*

Let there be an ellipse whose greater axis is AB, and focus S; and suppose a body acted upon by a centripetal force tending to the focus

focus S to revolve in the ellipse, it is required to find in AS the point E , such, that supposing a body to fall from A to E , it would acquire at E a velocity equal to the velocity of the body at the point A revolving in the ellipse.

Draw EF in any angle to AE ; and let EF be equal to ES ; join AF ; let CG be half the lesser axis; and let the rectangle contained by AC , SH , be equal to the square of CG ; let AK be an arc of the ellipse described in an indefinitely little part of time; draw KL perpendicular to AB , meeting AB in L ; complete the parallelogram $ALKD$; and draw LM parallel to EF , meeting AF in M .

Because the arc AK is described in an indefinitely little part of time, AL will be indefinitely little; therefore the rectangles ALB , LAB , may be considered as equal: but [Con. 2. 6.] the rectangle ALB is to the square of KL as the square of AC to the square of CG ; that is, as the square of AC to the rectangle contained by AC , SH ; that is, as AC to SH ; therefore the rectangle

angle

angle LAB is to the square of KL as AC to SH; that is, as AB to twice SH; that is, as the rectangle LAB to twice the rectangle contained by AL, SH; therefore twice the rectangle contained by AL, SH, is equal to the square of KL; therefore twice AL is to LK as LK to SH; and therefore twice AL is to LK in the subduplicate ratio of twice AL to SH; but twice AL is to AD, or LK, [10. Tr. 1.] as the velocity at L to the velocity in AD; that is, as the velocity that would be acquired in falling from A to L to the velocity that would be acquired in falling from A to E; therefore the velocity that would be acquired in falling from A to L is to the velocity that would be acquired in falling from A to E in the subduplicate ratio of twice AL to SH; but because the body revolving in the ellipse is acted upon by a centripetal force tending to the focus S, the centripetal force will [24. Tr. 1.] be reciprocally as the square of the distance from the focus S; therefore the velocity that would be acquired in falling from A to L is [2.] to the velocity that would be acquired in falling from A to E in the subduplicate ratio

Fig. 60.

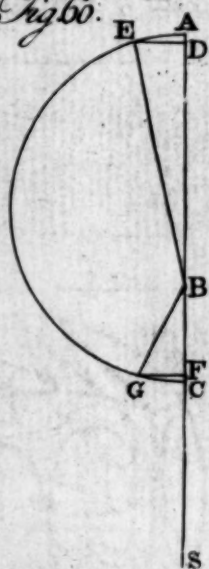


Fig. 61.

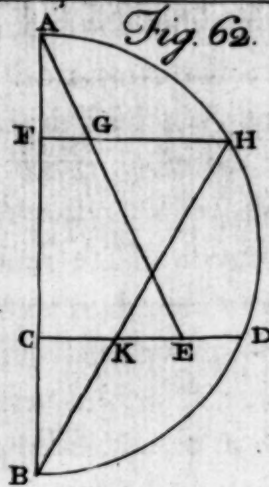


Fig. 62. Plak 9

Fig. 63.

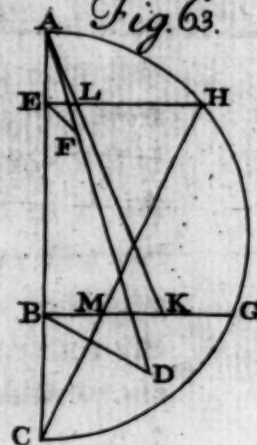


Fig. 66.

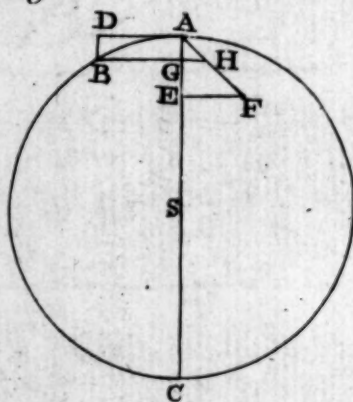


Fig. 65

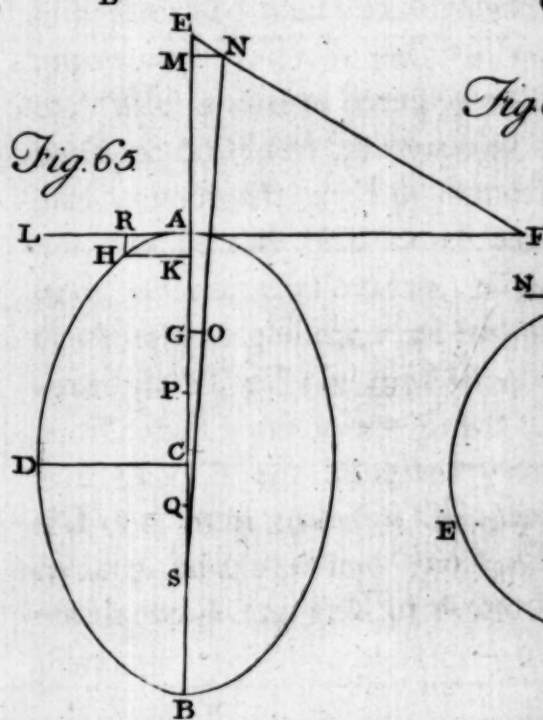
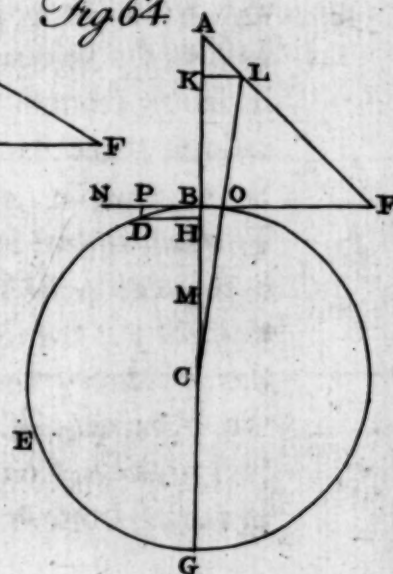


Fig. 64.



ratio of ML to LS ; that is, in the subduplicate ratio of LM to AS ; therefore the subduplicate ratio of twice AL to SH is equal to the subduplicate ratio of LM to AS ; therefore twice AL is to SH as LM to AS ; therefore, alternating, twice AL is to LM as HS to SA : but twice AL is to LM as twice AE to EF ; therefore twice AE is to EF as HS to SA : but because the rectangle contained by HS , AC , is equal to the square of CG ; that is, [Con. 2. 2.] equal to the rectangle ASB ; HS will be to SA as SB to AC ; therefore twice AE will be to EF as SB to AC ; that is, as twice SB to BA ; and therefore AE will be to EF , or ES , as SB to BA ; therefore SE will be to EA as AB to BS . Q. E. I.

P R O P. VII. Fig. 68.

Let there be an hyperbola whose transverse axis is AB , focus S , and centre C ; and suppose a body acted upon by a centripetal force tending to the focus S , to describe the hyperbola; it is required to find in AS the point E , such, that supposing a body to fall from A

C c

to

to E, it would acquire at E a velocity equal to the velocity of the body at the point A describing the hyperbola.

Draw EF in any angle to AE; and let EF be equal to ES; and join AF; let CG be half the second axis; and let the rectangle contained by AC, SH, be equal to the square of CG; let AK be an arc of the hyperbola described in an indefinitely little part of time; draw KL perpendicular to AB, meeting AB in L; complete the parallelogram ALKD; and draw LM parallel to EF, meeting AF in M.

Because the arc AK is described in an indefinitely little part of time, AL will be indefinitely little; therefore the rectangles ALB, LAB, may be considered as equal: but [Con. 3. 7.] the rectangle ALB is to the square of KL as the square of AC to the square of CG; that is, as the square of AC to the rectangle contained by AC, SH; that is, as AC to SH; therefore the rectangle LAB is to the square of KL as AC to SH; that is, as AB to twice SH; that is, as the rectangle LAB to twice the rectangle

angle

angle contained by AL , SH ; therefore twice the rectangle contained by AL , SH , is equal to the square of KL ; therefore twice AL is to KL as KL to SH ; therefore twice AL is to KL in the subduplicate ratio of twice AL to SH : but twice AL is [10. Tr. 1.] to KL , or AD , as the velocity at L to the velocity in AD ; that is, as the velocity that would be acquired in falling from A to L to the velocity that would be acquired in falling from A to E ; therefore the velocity that would be acquired in falling from A to L is to the velocity that would be acquired in falling from A to E in the subduplicate ratio of twice AL to SH : but because the body describing the hyperbola is acted upon by a centripetal force tending to the focus S , the centripetal force will [27. Tr. 1.] be reciprocally as the square of the distance from the focus S ; therefore the velocity that would be acquired in falling from A to L is [2.] to the velocity that would be acquired in falling from A to E in the subduplicate ratio of ML to LS ; that is, in the subduplicate ratio of LM to AS ; therefore the subduplicate

C c 2

ratio

ratio of twice AL to SH is equal to the subduplicate ratio of LM to AS; therefore twice AL is to SH as LM to AS; therefore, alternating, twice AL is to LM as HS to SA: but twice AL is to LM as twice AE to EF; therefore twice AE is to EF as SH to SA: but because the rectangle contained by SH, CA, is equal to the square of CG; that is, [Con. 3. 4.] equal to the rectangle ASB; HS will be to SA as SB to AC; that is, as twice SB to BA; and therefore AE will be to FE, or ES, as SB to BA; therefore, inversely, SE will be to EA as AB to BS. Q. E. I.

PROP. VIII. Fig. 69.

Let there be a parabola whose axis is AB, focus S, and vertex A; and suppose a body acted upon by a centripetal force tending to the focus S to describe the parabola; it is required to find in AS the point E, such, that supposing a body to fall from A to E, it would acquire at E a velocity equal to the velocity of the body at the point A describing the parabola.

Let

Let SC be equal to AS ; draw EF in any angle to AE ; let FE be equal to ES ; and join AF ; let AG be an arc of the parabola described in an indefinitely little part of time; draw GH perpendicular to AB , meeting AB in H ; complete the parallelogram $AHGD$; and draw HK parallel to EF , meeting AF in K .

Because the square of GH is [Con. 1. 12.] equal to four times the rectangle SAH ; that is, equal to twice the rectangle CAH ; therefore twice AH will be to HG as HG to CA ; therefore twice AH is to HG in the subduplicate ratio of twice AH to AC ; that is, in the subduplicate ratio of AH to AS : but because the arc AG is described in an indefinitely little part of time, twice AH will [10. Tr. 1.] be to GH , or AD , as the velocity at H to the velocity in AD ; that is, as the velocity that would be acquired in falling from A to H to the velocity that would be acquired in falling from A to E ; therefore the velocity that would be acquired in falling from A to H is to the velocity that would be acquired in falling from A to E in the subduplicate ratio of AH to AS : but because

cause the body describing the parabola is acted upon by a centripetal force tending to the focus S, the centripetal force will [28. Tr. 1.] be reciprocally as the square of the distance from the focus S; therefore the velocity that would be acquired in falling from A to H is [2.] to the velocity that would be acquired in falling from A to E in the subduplicate ratio of KH to HS; that is, in the subduplicate ratio of KH to AS; therefore AH, HK, are equal; and therefore AE is equal to EF; that is, equal to ES.

Q. E. I.

P R O P. IX. *Fig. 70.*

Let there be an ellipse whose greater axis is AB, focus S, and centre C; let D be any point in the ellipse; join SD; suppose a body acted upon by a centripetal force tending to the focus S to describe the ellipse: it is required to find in SD the point F, so, that supposing a body to fall from D to F, it would acquire at F a velocity equal to the velocity of the body at the point D describing the ellipse.

Draw

Draw FG in any angle to FS ; let FG be equal to FS ; and join DG ; join CD meeting the ellipse in K ; let CH be the semidiameter conjugate to CD ; and let the rectangle ACQ be equal to the square of CH ; let V be the other focus of the ellipse; and join DV ; produce SD to L , so that SL may be equal to AB ; let DE be a tangent to the ellipse at the point D , and DM an arc of the ellipse described in an indefinitely little part of time; draw MN parallel to DE , meeting SD , CD , in N , O ; complete the parallelogram $DNME$; draw NP parallel to FG , meeting DG in P ; and let CH meet SD in R .

Because the arc DM is described in an indefinitely little part of time, MN , MO , may be considered as equal, and the rectangles KDO , KOD , may likewise be considered as equal; therefore the square of MN will be to the rectangle KDO as the square of MO to the rectangle KOD ; that is, [Con. 2. 15.] as the square of CH to the square of CD : but the rectangle KDO is to the rectangle KDN as DO to DN ; that is, as CD to DR ; that is,

SD is to CQ as AB to DV ; and therefore SF is to FD as AB to DV. *Q. E. I.*

COR. If a body describe an ellipse, and be acted upon by a centripetal force tending to the focus S ; let D be any point in the ellipse, and V the other focus ; draw DS, DV ; in DS take the point F, so, that SF is to FD as the transverse axis is to DV ; the velocity of the body at D will be equal to the velocity that would be acquired in falling from D to F.

PROP. X. Fig. 71.

Let there be an hyperbola whose transverse axis is AB, focus S, and centre C ; and let D be any point in the hyperbola ; join SD ; and suppose a body acted upon by a centripetal force tending to the focus S to describe the hyperbola : it is required to find in SD the point F, so, that supposing a body to fall from D to F, it would acquire at F a velocity equal to the velocity of the body at the point D describing the hyperbola.

Draw

Draw FG in any angle to SF ; and let FG be equal to SF ; join DG , CD ; and produce CD to K ; let CD , CK , be equal; let CH be the semidiameter conjugate to CD ; and let the rectangle ACQ be equal to the square of CH ; let V be the other focus of the hyperbola; join DV ; produce DS to L ; let DV , DL , be equal; let DM be an arc of the hyperbola described in an indefinitely little part of time; draw MN parallel to DE a tangent to the hyperbola at D ; and let MN meet SD , CD , in N , O ; complete the parallelogram $DNME$; draw NP parallel to FG meeting DG in P ; and let CH meet SD in R .

Because the arc DM is described in an indefinitely little part of time, MN , MO , may be considered as equal; and the rectangles KDO , KOD , may likewise be considered as equal; therefore the square of MN will be to the rectangle KDO as the square of MO to the rectangle KOD ; that is, [Con. 3. 28.] as the square of CH to the square of CD : but the rectangle KDO is to the rectangle KDN as DO to DN ; that is, as DC to DR ; that is, as the

$D d 2$ square

square of CD to the rectangle CDR; therefore, by equality, the square of MN is to the rectangle KDN as the square of CH to the rectangle CDR: but because the square of CH is equal to the rectangle ACQ, and DR equal to AC, [Con. 5. 25.] therefore the square of CH is to the rectangle CDR as CQ to CD; that is, as twice CQ to KD; therefore the square of MN is to the rectangle KDN as twice CQ to KD; that is, as twice the rectangle contained by CQ, DN, to the rectangle KDN; therefore the square of MN is equal to twice the rectangle contained by CQ, DN; therefore twice DN is to NM as NM to CQ; and therefore twice DN will be to NM in the subduplicate ratio of twice DN to CQ: but twice DN is [10. Tr. 1.] to NM as the velocity at N to the velocity in DE; that is, as the velocity that would be acquired in falling from D to N to the velocity that would be acquired in falling from D to F; therefore the velocity that would be acquired in falling from D to N is to the velocity that would be acquired in falling from D to F in the subduplicate ra-

tio of twice DN to CQ : but because the
 body describing the hyperbola is acted upon
 by a centripetal force tending to the focus
 S, the centripetal force will [27. Tr. 1.] be
 reciprocally as the square of the distance
 from the focus S ; therefore the velocity
 that would be acquired in falling from D to
 N is [2.] to the velocity that would be ac-
 quired in falling from D to F in the subdu-
 plicate ratio of PN to NS, or SD ; there-
 fore the subduplicate ratio of twice DN to
 CQ is equal to the subduplicate ratio of
 NP to SD ; therefore twice DN is to CQ
 as NP to DS ; that is, as twice NP to
 twice DS ; therefore, alternating, twice
 DN is to twice NP as CQ to twice DS ;
 that is, DN is to NP as CQ to twice DS :
 but DN is to NP as DF to FG, or FS ;
 therefore DF is to FS as CQ to twice DS ;
 therefore, inversely, SF is to FD as twice
 DS to CQ : but [Con. 5. 30.] the rect-
 angle SDV is equal to the square of CH ;
 that is, equal to the rectangle ACQ ;
 therefore twice the rectangle SDV is equal
 to the rectangle contained by AB, CQ ;
 therefore twice SD is to CQ as AB to DV ;
 and

and therefore SF is to FD as AB to DV .

Q. E. I.

COR. If a body describe an hyperbola, and be acted upon by a centripetal force tending to the focus S ; let D be any point in the hyperbola, and V the other focus; draw DS , DV ; in DS take the point F , so, that SF is to FD as the transverse axis is to DV : the velocity of the body at D will be equal to the velocity that would be acquired in falling from D to F .

P R O P. XI. Fig. 72.

Let there be a parabola whose axis is AS , focus S , and vertex A ; and let D be any point in the parabola: suppose a body acted upon by a centripetal force tending to the focus S to describe the parabola: it is required to find in SD the point F , so, that supposing a body to fall from D to F , it would acquire at F a velocity equal to the velocity of the body at the point D describing the parabola.

Draw

Draw FG in any angle to FS , and let FG be equal to FS ; join DG ; draw CD parallel to AS ; let DE a tangent to the parabola at D meet AS in B ; produce DS to Q ; and let DS, SQ , be equal; let DM be an arc of the parabola described in an indefinitely little part of time; draw MN parallel to DE , meeting SD, CD , in N, O ; complete the parallelogram $DNME$; and draw NP parallel to FG , meeting DG in P .

Because BS, DO , are parallel, and likewise BD, NO , parallel, the triangles BSD, DNO , will be similar; therefore DO will be to DN as BS to SD : but because the angle CDS is [Con. 1. 9.] bisected by DB , and the angles CDB, DBS , are equal, because CD, BS ; are parallel; therefore the angles DBS, BDS , are equal; therefore BS is equal to SD ; therefore DN is equal to DO . Again, because the arc DM is described in an indefinitely little part of time, MN, MO , may be considered as equal; therefore the squares of MN, MO , will be equal: but the square of MO is [Con. 1. 13.] equal to the rectangle contained

tained by DO , and the parameter of the diameter CD ; that is, equal to four times the rectangle SDO , or SDN ; therefore the square of MN is equal to four times the rectangle SDN ; that is, equal to twice the rectangle QDN ; therefore twice DN is to NM as NM to DQ ; therefore twice DN is to NM in the subduplicate ratio of twice DN to DQ ; that is, in the subduplicate ratio of DN to DS : but twice DN is [10. Tr. 1.] to NM , or DE , as the velocity at N to the velocity in the tangent DE ; that is, as the velocity that would be acquired in falling from D to N to the velocity that would be acquired in falling from D to F ; therefore the velocity that would be acquired in falling from D to N is to the velocity that would be acquired in falling from D to F in the subduplicate ratio of DN to DS : but because the body describing the parabola is acted upon by a centripetal force tending to the focus S , the centripetal force will [28. Tr. 1.] be reciprocally as the square of the distance from the focus S ; therefore the velocity that would be acquired in falling from D to N is to the velocity that

that would be acquired in falling from D to F in the subduplicate ratio of PN to NS, or SD; therefore the subduplicate ratio of DN to DS is equal to the subduplicate ratio of PN to DS; therefore DN, NP, are equal; but DF is to FG, or FS, as DN to NP; therefore DF is equal to FS. Q. E. I.

COR. If a body describe a parabola, and be acted upon by a centripetal force tending to the focus S; let D be any point in the parabola; join SD; and bisect SD in F; the velocity of the body at D will be equal to the velocity that would be acquired in falling from D to F.

P R O P. XII. Fig. 67. 68. 69.

Supposing a body at the point A acted upon by a centripetal force that is reciprocally proportional to the square of its distance from the centre S, to be projected in the direction AD, perpendicular to AS, with a given velocity; it is required to determine the path of the body.

E e

Take

Take in AS the point E, such, that supposing the body to fall from A to E, it would acquire at E a velocity equal to the velocity with which the body is projected in the direction AD.

1st, Fig. 67. If SE be greater than EA, take in AS produced the point B, and let AB be to BS as SE to EA; with the transverse axis AB, and focus S, describe the ellipse AGB: this ellipse will [6.] be the path of the body.

2^d, Fig. 69. If SE be equal to EA, with the focus S, and vertex A, let there be a parabola described: this parabola will [8.] be the path of the body.

3^d, Fig. 68. If SE be less than EA, in SA produced take the point B, and let AB be to BS as SE to EA; with the transverse axis AB, and focus S, let there be an hyperbola described: this hyperbola will [7.] be the path of the body. Q. E. I.

P R O P.

P R O P. XIII. *Fig. 70. 71. 72.*

Supposing a body at the point D acted upon by a centripetal force that is reciprocally proportional to the square of its distance from the centre S, to be projected in the direction DE, not perpendicular to DS, with a given velocity; it is required to determine the path of the body.

Take in DS the point F, such, that supposing the body to fall from D to F, it would acquire at F a velocity equal to the velocity with which the body is projected in the direction DE; and produce ED to T.

1st, *Fig. 70.* If SF be greater than FD, draw DV on the same side of DE that DS is, making the angle VDT equal to the angle SDE; and in DV take the point V, such, that SD may be to DV as the excess of SF above FD to FD; with the foci S, V, let there be an ellipse described passing through the point D: this ellipse will [7.] be the path of the body.

E e 2

2d,

2d, *Fig. 72.* If SF be equal to FD , draw DC on the other side of DE that DS is, making the angle TDC equal to the angle TDS ; with the focus S , diameter DC , and vertex D , let there be a parabola described: this parabola will [8.] be the path of the body.

3d, *Fig. 71.* If SF be less than FD , draw DV on the other side DE that DS is, making the angle TDV equal to the angle TDS ; and in DV take the point V , so, that SD may be to DV as the excess of DF above FS to FD ; with the foci V, S , let there be an hyperbola described: this hyperbola will [7.] be the path of the body.
Q. E. I.

P R O P. XIV. *Fig. 73.*

Let there be an ellipse whose greater axis is AB , and focus S ; draw SC perpendicular to AB , meeting the ellipse in C ; suppose a body acted upon by a centripetal force tending to the focus S to describe the ellipse; let L be any point in the ellipse; join SL :
the

the centripetal force at L is to the centrifugal force at the same point L as SL to SC.

Let AD be described in an indefinitely little part of time; draw DE perpendicular to AB, meeting AB in E; complete the parallelogram AEDF; join SF; with the centre S, and distance SA, describe a circle meeting SF in G, H; let CS meet the ellipse in K; and let M be the centre of the ellipse; draw MN perpendicular to AB, meeting the ellipse in N.

Because DE, AF, are equal, the square of DE will be equal to the square of AF; that is, (because of the circle), equal to the rectangle HFG; therefore the rectangle AEB will be to the rectangle HFG as the rectangle AEB to the square of DE; that is, [Con. 2. 6.] as the square of AM to the square of MN: but because CK is [Con. 5. 24.] equal to the parameter of AB, the square of AM is to the square of MN as AB to CK; therefore the rectangle AEB will be to the rectangle HFG as AB to CK. Because AD is indefinitely little, the rectangles BAE, AEB, may be considered as equal,

equal, and likewise the rectangles HGF, HFG, may be considered as equal; therefore the rectangle BAE will be to the rectangle HGF as AB to CK; that is, as the rectangle BAE to the rectangle contained by CK, AE; therefore the rectangle contained by CK, AE, is equal to the rectangle HGF; and therefore the rectangle contained by CS, AE, is equal to the rectangle SGF; therefore AE is to FG as SG, or SA, to SC: but AE and FG are the effects produced by the centripetal and centrifugal forces in the same time; therefore the centripetal force at A is to the centrifugal force at A as AS to SC.

Again, because [24. Tr. I.] the centripetal force at L is to the centripetal force at A as the square of AS to the square of SL; that is, as the cube of AS to the parallelopiped whose base is the square of SL, and altitude AS; and the centripetal force at A is to the centrifugal force at A as AS to SC; that is, as the parallelopiped whose base is the square of SL, and altitude AS, to the parallelopiped whose base is the square of SL, and altitude SC; therefore,
by

by equality, the centripetal force at L will be to the centrifugal force at A as the cube of AS to the parallelopiped whose base is the square of SL, and altitude SC: but the centrifugal force at A is [8. Tr. 1.] to the centrifugal force at L as the cube of SL to the cube of SA; therefore, by perturbate equality, the centripetal force at L will be to the centrifugal force at L as the cube of SL to the parallelopiped whose base is the square of SL, and altitude SC; that is, as SL to SC. *Q. E. D.*

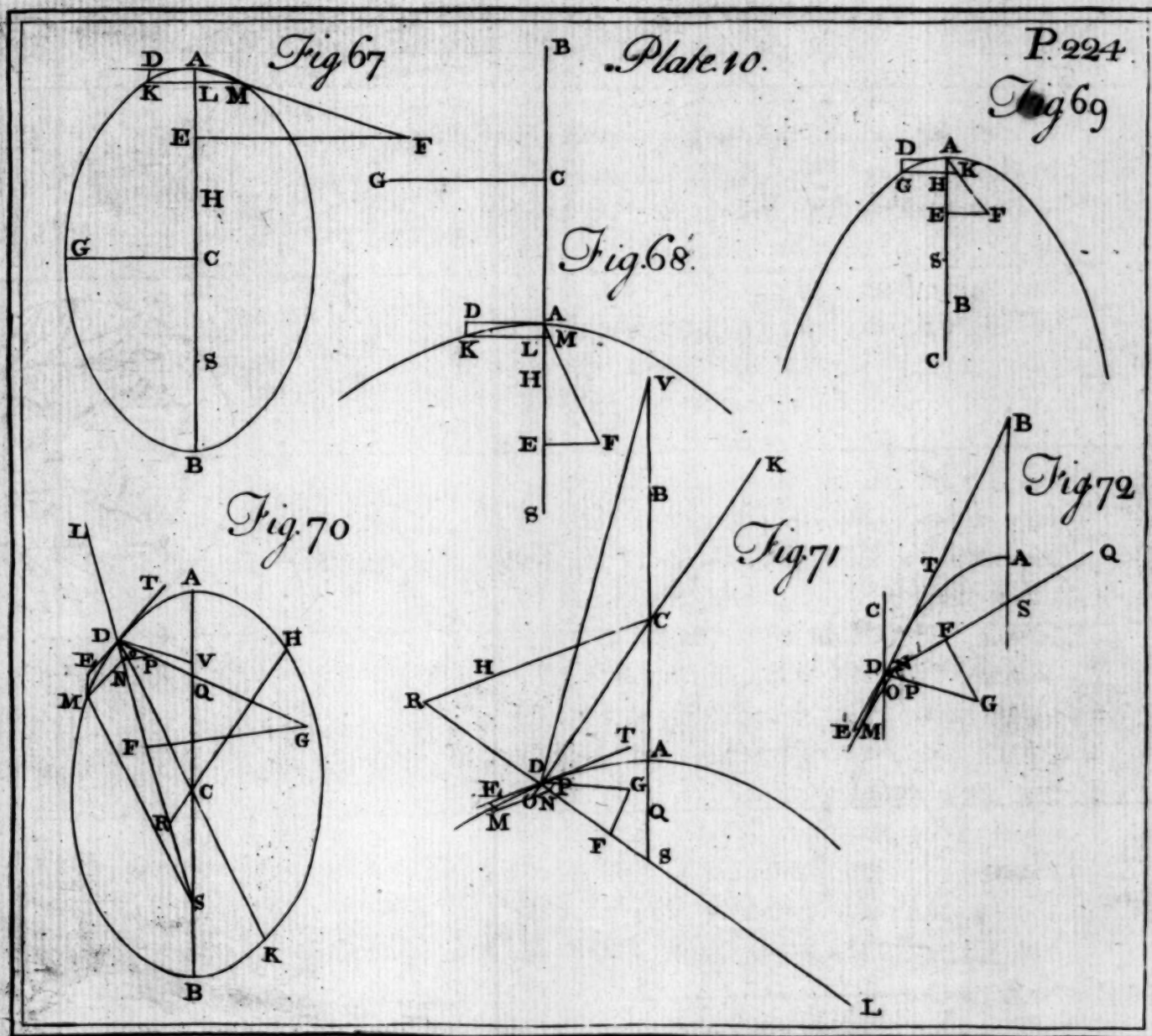
PROP. XV. *Fig. 74.*

Let there be a parabola ABC whose axis is AS, and focus S; draw SB perpendicular to AS, meeting the parabola in B; supposing a body acted upon by a centripetal force tending to the focus S to describe the parabola, the centripetal force at any point C in the parabola is to the centrifugal force at the same point C as CS to SB.

Let SB meet the parabola again in D, and let AE be a part of the parabola described

scribed in an indefinitely little part of time; draw EF perpendicular to AS , meeting AS in F ; and complete the parallelogram $AFEG$; join SG ; with the centre S , and distance SA , describe a circle meeting SG in H , K .

Because EF , AG , are equal, the square of EF will be equal to the square of AG ; that is, because of the circle, equal to the rectangle KGH ; but the square of EF is [Con. 1. 12.] equal to four times the rectangle SAF ; therefore the rectangle KGH is equal to four times the rectangle SAF : but because BD is [Con. 5. 24.] equal to four times SA ; therefore the rectangle contained by BD , AF , is equal to four times the rectangle SAF ; therefore the rectangle KGH is equal to the rectangle contained by BD , AF : and because AE is indefinitely little, the rectangles KGH , KHG , may be considered as equal; therefore the rectangle contained by BD , AF , is equal to the rectangle KHG ; and therefore the rectangle contained by SB , AF , is equal to the rectangle SHG ; therefore AF is to GH as SH , or SA , to SB : but AF
and



and GH are the effects produced by the centripetal and centrifugal forces in the same time; therefore the centripetal force at A is to the centrifugal force at the same point A as AS to SB.

Again, because [28. Tr. 1.] the centripetal force at C is to the centripetal force at A as the square of AS to the square of SC; that is, as the cube of AS to the parallelopiped whose base is the square of SC, and altitude AS; and the centripetal force at A is to the centrifugal force at the same point A as AS to SB; that is, as the parallelopiped whose base is the square of SC, and altitude AS, to the parallelopiped whose base is the square of SC, and altitude SB; therefore, by equality, the centripetal force at C will be to the centrifugal force at A as the cube of AS to the parallelopiped whose base is the square of SC, and altitude SB: but the centrifugal force at A is to the centrifugal force at C [8. Tr. 1.] as the cube of SC to the cube of SA; therefore, by perturbate equality, the centripetal force at C is to the centrifugal force at C as the cube of CS to the parallelopiped

F f

whose

whose base is the square of SC , and altitude SB ; that is, as CS to SB . *Q. E. D.*

P R O P. XVI. *Fig. 75.*

Let there be an hyperbola ABC whose transverse axis is AD, focus S, and centre F; draw SB perpendicular to AS, meeting the hyperbola in B; supposing a body acted upon by a centripetal force tending to the focus S to describe the hyperbola, the centripetal force at any point C in the hyperbola will be to the centrifugal force at the same point C as CS to SB.

Let BS meet the hyperbola again in E ; let FG be half the second axis; and let AH be a part of the hyperbola described in an indefinitely little part of time; draw HK perpendicular to AS , meeting AS in K ; and complete the parallelogram $AKHL$; join SL ; with the centre S , and distance SA , describe a circle meeting SL in M, N .

Because HK, AL , are equal, the square of HK will be equal to the square of AL ; that is, equal to the rectangle MLN ; therefore

therefore the rectangle AKD will be to the rectangle MLN as the rectangle AKD to the square of HK; that is, [Con. 3. 7.] as the square of the AF to the square of FG: but because [Con. 5. 24.] BE is equal to the parameter of AD, the square of AF is to the square of FG as AD to BE; therefore the rectangle AKD is to the rectangle MLN as AD to BE: but because AH is indefinitely little, the rectangles AKD, DAK, may be considered as equal; likewise the rectangles MLN, NML, may be considered as equal; therefore the rectangle DAK will be to the rectangle NML as DA to BE; that is, as the rectangle DAK to the rectangle contained by BE, AK; therefore the rectangle NML is equal to the rectangle contained by BE, AK; and therefore the rectangle SML will be equal to the rectangle contained by BS, AK; therefore AK is to LM as MS, or SA, to BS: but AK and ML are the effects produced by the centripetal and centrifugal forces in the same time; therefore the centripetal force at A is to the cen-

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trifugal force at the same point A as AS to SB.

Again, because [27. Tr. 1.] the centripetal force at C is to the centripetal force at A as the square of AS to the square of SC; that is, as the cube of AS to the parallelopiped whose base is the square of SC, and altitude AS; and the centripetal force at A is to the centrifugal force at A as AS to SB; that is, as the parallelopiped whose base is the square of SC, and altitude AS, to the parallelopiped whose base is the square of SC, and altitude SB; therefore, by equality, the centripetal force at C will be to the centrifugal force at A as the cube of AS to the parallelopiped whose base is the square of SC, and altitude SB: but the centrifugal force at A is [8. Tr. 1.] to the centrifugal force at C as the cube of SC to the cube of AS; therefore, by perturbate equality, the centripetal force at C is to the centrifugal force at C as the cube of SC to the parallelopiped whose base is the square of SC, and altitude SB; that is, as SC to SB. Q. E. D.

P R O P.

P R O P. XVII. Fig. 76.

Let there be an ellipse whose centre is C ; suppose a body acted upon by a centripetal force tending to the centre C to describe the ellipse: it is required to find the height from which the body would fall, so as to acquire a velocity equal to the velocity of the body at the point A in the ellipse.

Join CA , meeting the ellipse in E ; and let CD be the femidiameter conjugate to CA ; let AF be a tangent to the ellipse at the point A , and AG a part of the ellipse described in an indefinitely little part of time; draw GH parallel to AF , meeting CA in H ; and complete the parallelogram $AHGF$; draw AK perpendicular to CA ; and join CK ; let AB be the height from which the body would fall, so as to acquire at the point A a velocity equal to the velocity at the point A in the ellipse; and draw BL , HM , parallel to AK , meeting CK in L , M .

Because the body describing the ellipse is acted

acted upon by a centripetal force tending to the centre of the ellipse, the centripetal force will [25. Tr. 1.] be directly as the distance of the body from the centre; therefore the centripetal force at B will be to the centripetal force at A as BL to AK; therefore the velocity that would be acquired in falling from B to A will [3. Tr. 1.] be to the velocity that would be acquired in falling from A to H in the subduplicate ratio of the space LBAK to the space KAHM; therefore the velocity of the body at the point A in the ellipse will be to the velocity that would be acquired in falling from A to H in the subduplicate ratio of the space LBAK to the space KAHM: but the velocity of the body at the point A in the ellipse is [10. Tr. 1.] to the velocity that would be acquired in falling from A to H as AF to twice AH; therefore AF is to twice AH in the subduplicate ratio of the space LBAK to the space KAHM; therefore the space LBAK will be to the space KAHM as the square of AF to four times the square of AH; therefore, alternating, the space LBAK will be to the square of AF

as the space $KAHM$ to four times the square of AH : but the space $KAHM$ is to four times the square of AH as the rectangle KAH to four times the square of AH ; that is, as KA to four times AH ; that is, as the rectangle KAE to four times the rectangle EAH , or EHA ; therefore the space $LBAK$ will be to the square of AF as the rectangle KAE to four times the rectangle EHA ; therefore, alternating, the space $LBAK$ will be to the rectangle KAE as the square of AF to four times the rectangle EHA : but the rectangle KAE is quadruple of the triangle KAC ; therefore the space $LBAK$ will be to the triangle KAC as the square of AF , or GH , to the rectangle EHA : but the square of GH is to the rectangle EHA [Con. 2. 15.] as the square of CD to the square of CA ; therefore the space $LBAK$ will be to the triangle KAC as the square of CD to the square of CA : but because the triangle LBC is to the triangle KAC as the square of CB to the square of CA ; by division, the space $LBAK$ will be to the triangle KAC as the rectangle ABE to the square of AC ; therefore

fore the rectangle ABE is to the square of AC as the square of CD to the square of AC; and therefore the rectangle ABE is equal to the square of CD. Q. E. I.

P R O P. XVII. Fig. 76.

Suppose a body at the point A, acted upon by a centripetal force that is as its distance from the centre C, to be projected in the direction AF with a given velocity, it is required to determine the path of the body.

Join AC; produce AC to E; and let CE be equal to AC; in CA produced take AB so, that supposing the body to fall from B to A, it would acquire at A a velocity equal to the velocity with which the body is projected in the direction AF; draw CD parallel to AF; and let the square of CD be equal to the rectangle ABE; with the conjugate semidiameters CA, CD, let there be an ellipse described: this ellipse will [17.] be the path of the body. Q. E. I.

P R O P.

PROP. XIX. Fig. 77.

Let there be an ellipse whose centre is C ; let CA , CB , be the two semiaxes; suppose a body acted upon by a centripetal force tending to the centre C to describe the ellipse, the centrifugal force at the point A will be to the centripetal force at the point A in the duplicate ratio of CB to CA .

Let AC meet the ellipse in D ; produce CA to F so, that the rectangle AFD may be equal to the square of CB ; draw AE perpendicular to CA ; join CE ; and draw FG parallel to AE , meeting CE in G .

Because the body describing the ellipse is acted upon by a centripetal force tending to the centre C , the centripetal force will [25. Tr. 1.] be as the distance of the body from the centre C ; therefore the centripetal force at A will be to the centripetal force at F as AE to FG : and because the velocity of the body at the point A describing the ellipse is [17.] equal to the velocity that would be acquired in falling

Gg from

from F to A; therefore [II. Tr. I.] the centrifugal force at A will be to the centripetal force at A as twice the space AFGE to the rectangle CAE: but the rectangle CAE is double of the triangle CAE; therefore the centrifugal force at A will be to the centripetal force at A as the space AFGE to the triangle CAE: but the space AFGE is to the triangle CAE as the rectangle AFD to the square of CA; that is, as the square of CB to the square of CA; therefore the centrifugal force at A will be to the centripetal force at A as the square of CB to the square of CA; and therefore the centrifugal force at A will be to the centripetal force at A in the duplicate ratio of BC to CA. Q. E. D.

COR. Let there be an ellipse whose centre is C; and let CA, CB, be the two semi-axes; suppose a body acted upon by a centripetal force tending to the centre C, that is as its distance from the centre, to be projected in the direction AH perpendicular to CA; and let the centrifugal force at the point A, arising from the projectile force

in

in the direction AH, be to the centripetal force at the point A in the duplicate ratio of BC to CA: the body will describe the ellipse.

PROP. XX. Fig. 78.

In the straight line AS let S, B, be two given points; and let the point S be between the points A, B; let CDE, FGH, be two curves, such, that if from any two points A, K, in AS, there be drawn AC, KD, perpendicular to AS, meeting the curve CDE in C, D, and likewise meeting the curve FGH in F, G, the square of AS is to the square of SK as KD to AC, likewise the rectangle BAC is to the rectangle BKD as the rectangle SAF to the rectangle SKG; in AF take AL, such, that FA is to AL as AB to BS; and produce FA to M, such, that FA is to AM as BA to AS; and let LNO, MPQ, be two curves, such, that if from any point K in AS there be drawn KN perpendicular to AS, meeting the curves LNO, MPQ, in N, P, the cube of SK is to the cube of SA as AL to

G g 2

KN;

KN; likewise the square of KS is to the square of SA as AM to KP: if from any point K in AS there be drawn KG perpendicular to AS, meeting the curves FGH, LNO, MPQ, in G, N, P, PN will be equal to KG.

Because AM is to KP as the square of SK to the square of SA; that is, as AC to KD, AM will be to AC as KP to KD; therefore the rectangle BAM will be to the rectangle BAC as the rectangle BKP to the rectangle BKD; and alternating, the rectangle BAM will be to the rectangle BKP as the rectangle BAC to the rectangle BKD; that is, as the rectangle SAF to the rectangle SKG: but because FA is to AM as BA to AS, the rectangle BAM will be equal to the rectangle SAF; therefore the rectangle BKP will be equal to the rectangle SKG; therefore GK will be to KP as BK to KS. Because FA is to AM as BA to AS, AM will be to AF as SA to AB: but AF is to AL as AB to BS; therefore AM is to AL as AS to SB. Because KP is to AM as the square of SA to the square of SK;

SK; that is, as the cube of SA to the parallelopiped whose base is the square of SK, and altitude AS; and AM is to AL as AS to SB; that is, as the parallelopiped whose base is the square of SK, and altitude AS, to the parallelopiped whose base is the square of SK, and altitude SB; therefore KP will be to AL as the cube of AS to the parallelopiped whose base is the square of SK, and altitude SB: but AL is to KN as the cube of SK to the cube of AS; therefore KP is to KN as the cube of SK to the parallelopiped whose base is the square of SK, and altitude SB; that is, as KS to SB. Because GK is to KP as BK to KS, and KP to KN as KS to SB; therefore GK is to KN as BK to BS: but because GK is to KP as BK to KS, and GK is to KN as BK to BS, and BK is equal to the sum of KS, SB; therefore PN is equal to KG. Q. E. D.

COR. From this it is evident, that the area FAKG is equal to the sum of the areas MAKP, LAKN.

PROP.

PROP. XXI. Fig. 79.

In the straight line AS let S, B, be two given points; and let the point S be between the points A, B; let CDE, FGH, be two curves, such, that if from any two points A, K, in AS there be drawn AC, KD, perpendicular to AS, meeting the curve CDE in C, D, and likewise meeting the curve FGH in F, G, the square of AS is to the square of SK as KD to AC, likewise the rectangle BAC is to the rectangle BKD as the rectangle SAF to the rectangle SKG; in BS take the point L, such, that BL is to LS as BA to AS; and let the rectangle MAL be equal to the square of AS; bisect AL in b; from any point K in AS draw KG perpendicular to AS, meeting the curve FGH in G: the area FAKG will be to the rectangle AKL as the rectangle MAF to the square of KS, likewise the area FAKG. will be to the rectangle AKL as the rectangle SKG to twice the rectangle contained by BK, Sb.

In

In FA take AN such, that FA is to AN as AB to BS; produce FA to O; and let FA be to AO as BA to AS; let NPQ, OTV, be two curves, such, that if from any point K in AS there be drawn KP perpendicular to AS, meeting the curves NPQ, OTV, in P, T, the cube of SK is to the cube of SA as AN to KP, and likewise the square of SK is to the square of SA as AO to KT; produce AS to X; and let AS, SX, be equal; join XN, SO; let XN meet KP in Y, and SO meet KT in Z; draw La perpendicular to AL meeting XN in a.

Because [18. Tr. 1.] the rectangle SAN is to the area NAKP as twice the square of KS to the rectangle AKX; inversely, the area NAKP will be to the rectangle SAN as the rectangle AKX to twice the square of KS; therefore, alternating, the area NAKP will be to the rectangle AKX as the rectangle SAN to twice the square of KS: but the rectangle AKX is to the rectangle AKY as XK to KY; that is, as XA to AN; that is, as the rectangle XAS to the rectangle SAN; therefore, by perturbate equality, the

the area $NAKP$ will be to the rectangle AKY as the rectangle XAS to twice the square of KS ; that is, as the square of AS to the square of KS ; therefore the area $NAKP$ will be to the square of AS as the rectangle AKY to the square of KS . Again, because the square of SA is to the rectangle SAO as SA to AO ; that is, as SK to KZ ; that is, as the square of SK to the rectangle SKZ ; and the rectangle SAO is [11. Tr. 2.] to the area $OAKT$ as SK to KA ; that is, as the rectangle SKZ to the rectangle AKZ ; the square of SA will be to the area $OAKT$ as the square of SK to the rectangle AKZ ; but because the area $NAKP$ is to the square of AS as the rectangle AKY to the square of KS ; and the square of AS is to the area $OAKT$ as the square of SK to the rectangle AKZ ; therefore, by equality, the area $NAKP$ will be to the area $OAKT$ as the rectangle AKY to the rectangle AKZ ; that is, as KY to KZ ; therefore the sum of the areas $NAKP$, $OAKT$, will be to the area $OAKT$ as YZ to KZ ; but the area $FAKG$ is [cor. to prop. 20.] equal to the sum of the areas $NAKP$, $OAKT$;
therefore

Fig. 73 Plate. 11. Fig 74

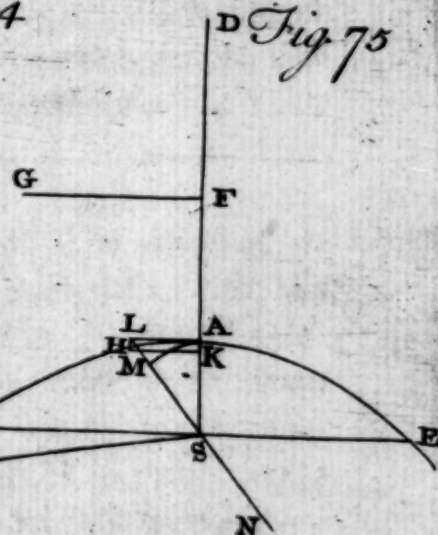
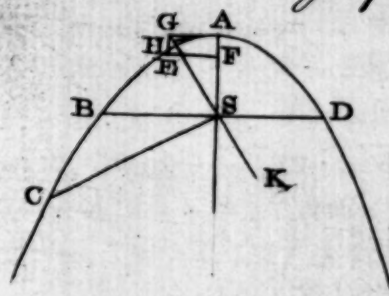
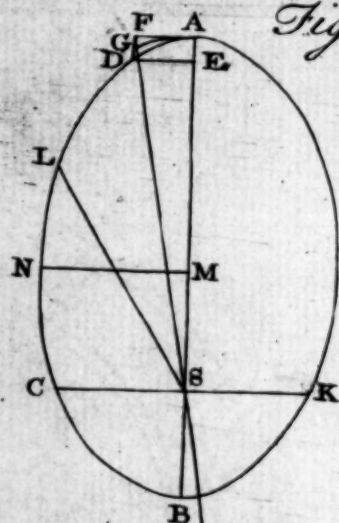


Fig. 76

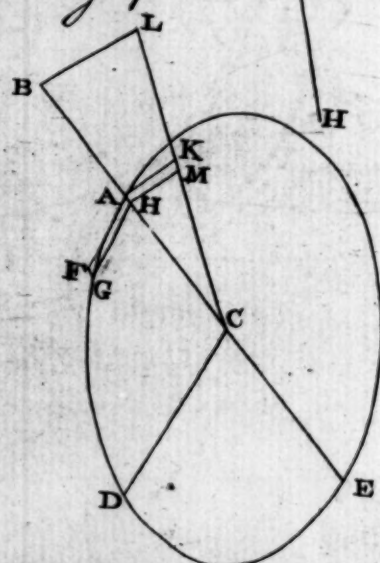


Fig. 77

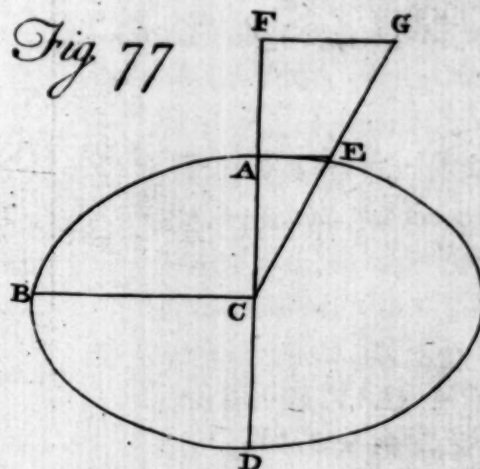
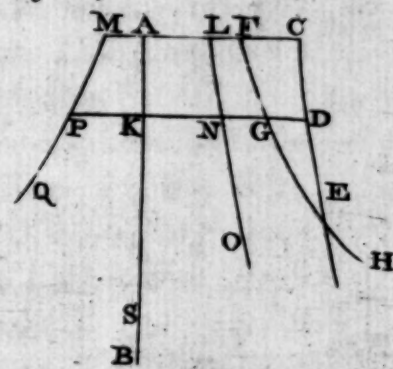


Fig. 78



therefore the area FAKG is to the area OAKT as YZ to ZK; that is, as the rectangle contained by YZ, AK, to the rectangle AKZ; therefore, alternating, the area FAKG will be to the rectangle contained by YZ, AK, as the area OAKT to the rectangle AKZ: but because the square of AS is to the area OAKT as the square of KS to the rectangle AKZ; alternating, the square of AS will be to the square of SK as the area OAKT to the rectangle AKZ; therefore the area FAKG will be to the rectangle contained by YZ, AK, as the square of AS to the square of SK.

Again, because BL is to LS as BA to AS, BL will be to BA as LS to SA; therefore inversely, AB is to BL as AS to SL; therefore [26. Tr. 2.] because AL is bisected in *b*, the rectangle B*b*S is equal to the square of *b*L; therefore the rectangle BS*b* is equal to the rectangle LSA; therefore AS is to S*b* as SB to SL: but because AX is double of AS, and AL double of A*b*; therefore XL will be double of S*b*; therefore AX will be to XL as AS to S*b*; that is, as BS to SL: but AN is to La as AX to XL; therefore

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AN

AN is to La as BS to SL ; therefore, inversely, La is to AN as SL to SB : but because FA is to AN as AB to BS , AN will be to AF as BS to AB ; and AF is to AO as AB to AS ; therefore AN is to AO as BS to SA : but because La is to AN as SL to SB , and AN is to AO as BS to SA ; therefore La is to AO as LS to SA ; therefore OSa is a straight line; therefore YZ is to NO as Ya to aN ; that is, as LK to LA ; that is, because NO is equal to AF , YZ will be to AF as LK to LA ; therefore, alternating, YZ will be to LK as AF to LA ; therefore the rectangle contained by YZ , AK , will be to the rectangle AKL as AF to LA ; that is, as the rectangle MAF to the rectangle MAL : but the rectangle MAL is equal to the square of AS ; therefore the rectangle contained by YZ , AK , will be to the rectangle AKL as the rectangle MAF to the square of AS : but because the area $FAKG$ is to the rectangle contained by YZ , AK , as the square of AS to the square of SK ; and the rectangle contained by YZ , AK , is to the rectangle AKL as the rectangle MAF to the square

square of AS; therefore, by perturbate equality, the area FAKG will be to the rectangle AKL as the rectangle MAF to the square of SK. *Q. E. D.*

Again, in AS take Sd equal to SL. Because BA is to AS as BL to LS; that is, as BL to Sd , BA will be to BL as AS to Sd ; therefore, conversely, BA will be to AL as AS to Ad . Because the square of SK is to the square of AS as AC to KD, and the rectangle MAL is equal to the square of AS, the square of SK will be to the rectangle MAL as AC to KD; that is, as the rectangle CAL to the rectangle contained by KD, AL, and the rectangle MAL is to the rectangle MAF as AL to AF; that is, as the rectangle contained by KD, AL, to the rectangle contained by KD, AF, the square of SK will be to the rectangle MAF as the rectangle CAL to the rectangle contained by KD, AF. Because the rectangle BAC is to the rectangle BKD as the rectangle SAF to the rectangle SKG; inversely, the rectangle BKD will be to the rectangle BAC as the rectangle SKG to the rectangle SAF: but the rectangle BAC is to

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the

the rectangle CAL as BA to AL; that is, as AS to Ad; that is, as the rectangle SAF to the rectangle dAF; therefore, by equality, the rectangle BKD will be to the rectangle CAL as the rectangle SKG to the rectangle dAF; therefore, inversely, the rectangle CAL will be to the rectangle BKD as the rectangle dAF to the rectangle SKG; but the rectangle BKD is to the rectangle contained by KD, AF, as BK to AF; that is, as the rectangle contained by BK, Ad, to the rectangle dAF; therefore, by perturbate equality, the rectangle CAL is to the rectangle contained by KD, AF, as the rectangle contained by BK, Ad, to the rectangle SKG; but the square of SK is to the rectangle MAF as the rectangle CAL to the rectangle contained by KD, AF; therefore the square of SK is to the rectangle MAF as the rectangle contained by BK, Ad, to the rectangle SKG; therefore, inversely, the rectangle MAF is to the square of SK as the rectangle SKG to the rectangle contained by BK, Ad. Because AL is double of Lb, and Ld double of LS, Ad will be double of Sb; therefore the rectangle contained by BK, Ad is double

ble of the rectangle contained by BK, Sb ; therefore the rectangle MAF is to the square of SK as the rectangle SKG to twice the rectangle contained by BK, Sb . But the area FAKG is to the rectangle AKL as the rectangle MAF to the square of SK; therefore the area FAKG is to the rectangle AKL as the rectangle SKG to twice the rectangle contained by BK, Sb , Q. E. D.

P R O P. XXII. Fig. 80.

In the straight line AS let S, B, be two given points; and let the point S be between the points A, B; let CDE, FGH, be two curves, such, that if from any two points A, K, in AS, there be drawn AC, KD, perpendicular to AS, meeting the curve CDE in C, D, and likewise meeting the curve FGH in F, G: the square of AS is to the square of SK as KD to AC; likewise the rectangle BAC is to the rectangle BKD as the rectangle SAF to the rectangle SKG: in BS take the point L, such, that BL is to LS as BA to AS; upon AL let there
be

be a circle AML described; from any point K in AS draw KG perpendicular to AS , meeting the curve FGE in G , and the circle in M ; join SM , meeting AF in N , and the circle in O : the area $FAKG$ will be to the square of AN as AF to AL ; likewise the area $FAKG$ will be to the rectangle SKG as the square of KM to the rectangle SMO .

Let the rectangle PAL be equal to the square of AS ; and bisect AL in Q .

Because the area $FAKG$ is [21.] to the rectangle AKL as the rectangle PAF to the square of KS , alternating, the area $FAKG$ will be to the rectangle PAF as the rectangle AKL to the square of KS : but the rectangle AKL is equal to the square of KM , because of the circle; therefore the area $FAKG$ will be to the rectangle PAF as the square of KM to the square of KS ; that is, as the square of AN to the square of AS ; therefore, alternating, the area $FAKG$ will be to the square of AN as the rectangle PAF to the square of AS : but because the rectangle PAL is equal to the square of AS ,
the

the rectangle PAF will be to the square of AS as the rectangle PAF to the rectangle PAL; that is, as AF to AL; therefore the area FAKG will be to the square of AN as AF to AL. Q. E. D.

Again, because [21.] the area FAKG is to the rectangle AKL as the rectangle SKG to twice the rectangle contained by BK, SQ, alternating, the area FAKG will be to the rectangle SKG as the rectangle AKL to twice the rectangle contained by BK, SQ. But because AL is bisected in Q, the rectangle BQS is [26. Tr. 2.] equal to the square of LQ; therefore [5. Tr. 2.] the rectangle SMO is equal to twice the rectangle contained by BK, SQ; and therefore because the rectangle AKL is equal to the square of KM, the rectangle AKL will be to twice the rectangle contained by BK, SQ, as the square of KM to the rectangle SMO; therefore the area FAKG is to the rectangle SKG as the square of KM to the rectangle SMO. Q. E. D.

COR. 1. From any point R in AS draw RT perpendicular to AS, meeting the semi-circle

circle AML in T, and join ST, meeting KM in V. Suppose a body acted upon by a centripetal force tending to the centre S to descend in the straight line AS, and the centripetal force at any two points A, K, to be as AF to KG; the velocity at K that would be acquired by falling from A to K will be to the velocity at R that would be acquired by falling from A to R as MK to KV.

Let RT meet the curve FGH in X; and let ST meet AF in Y. Because the area FAKG is to the square of AN as AF to AL, and the area FARX is to the square of AY as AF to AL, the area FAKG will be to the square of AN as the area FARX is to the square of AY; therefore, alternating, the area FAKG will be to the area FARX as the square of AN to the square of AY; that is, as the square of KM to the square of KV; therefore KM is to KV in the subduplicate ratio of the area FAKG to the area FARX: but [3. Tr. 1.] the velocity at K is to the velocity at R in the subduplicate ratio of the area FAKG to the area FARX;

therefore

therefore the velocity at K is to the velocity at R as MK to KV. *Q. E. D.*

COR. 2. Suppose a body at the point K, to be projected in the direction KZ, perpendicular to AK, with the velocity that would be acquired by falling from A to K; the centrifugal force at the point K arising from the projectile force in the direction KZ will be to the centripetal force at the point K as twice the square of KM to the rectangle SMO.

Because the area FAKG is to the rectangle SKG as the square of KM to the rectangle SMO, twice the area FAKG will be to the rectangle SKG as twice the square of KM to the rectangle SMO: but the centrifugal force at the point K, arising from the projectile force in the direction KZ, is [II.Tr.I.] to the centripetal force at the point K as twice the area FAKG to the rectangle SKG; therefore the centrifugal force at the point K, arising from the projectile force in the direction KZ, is to the centripetal force at the point K as twice the square of KM to the rectangle SMO. *Q. E. D.*

I i P R O P.

P R O P. XXIII. Fig. 81.

In the straight line SP, let S, D, be two given points, and let the point S be between the points P, D; let EFG, HKG, be two curves, such, that if from any two points A, P, in AS, there be drawn AF, PE, perpendicular to AS, meeting the curve EFG, in F, E, and likewise meeting the curve HKE in K, H, the square of SA is to the square of SP as PE to AF, likewise the rectangle DAF is to the rectangle DPE as the rectangle SAK to the rectangle SPH; bisect AP in C; and in CS take CB, such, that the rectangle SCB is equal to the square of CA; let the curves EFG, HKG, intersect each other in the point G; and draw GL perpendicular to AS, meeting AS in L: the area EPAF will be to the area HPAK as the rectangle contained by DL, SB, to the rectangle contained by SL, DB.

In CD, take CM, such, that the square of CM is equal to the rectangle SCD; with the

the centre C, and distance CM, describe a circle, meeting CP in N, and AF, PE, in O, V; join SO, SV, meeting the circle in Q, R; and draw NX perpendicular to CN, meeting the curves EFG, HKG, in X, T; and let the rectangle SPY be equal to the rectangle PAS.

Because the square of CM is equal to the rectangle SCD, and the square of CA equal to the rectangle SCB, the rectangle MAN will be equal to the rectangle contained by DB, SC; that is, the square of AO will be equal to the rectangle contained by DB, SC; and because the rectangle SOQ is [5. Tr. 2.] equal to twice the rectangle contained by DA, SC; therefore the square of AO will be to the rectangle SOQ as the rectangle contained by DB, SC, to twice the rectangle contained by DA, SC; that is, as DB to twice DA. The same way it is shown, that the square of PV is to the rectangle SVR as DB to twice DP: but the area TNAK is [22.] to the rectangle SAK as the square of AO to the rectangle SOQ; therefore the area TNAK will be to the rectangle SAK as DB to twice DA;

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that

that is, as the rectangle contained by DB, AF, to twice the rectangle DAF: but the rectangle SAK is to the rectangle SPH as the rectangle DAF to the rectangle DPE; that is, as twice the rectangle DAF to twice the rectangle DPE; therefore the area TNAK is to the rectangle SPH as the rectangle contained by DB, AF, to twice the rectangle DPE: but because the area TNPH is [22.] to the rectangle SPH as the square of PV to the rectangle SVR; that is, as DB to twice DP; inversely, the rectangle SPH is to the area TNPH as twice DP to DB; that is, as twice the rectangle DPE to the rectangle contained by DB, PE; therefore the area TNAK is to the area TNPH as the rectangle contained by DB, AF, to the rectangle contained by DB, PE; that is, as AF to PE; that is, as the square of SP to the square of SA; therefore the area TNAK will be to the area HPAK as the square of SP to the excess of the square of SP above the square of SA; that is, as the square of SP to twice the rectangle contained by SC, AP: but because the rectangle SCB is equal to the square

square of AC, the rectangle CSB will be equal to the rectangle ASP; therefore PS will be to SB as SC to CA: but because the square of PS is to the rectangle PSB as PS to SB, and the rectangle contained by SC, AP, is to the rectangle SAP as SC to CA; therefore the square of PS will be to the rectangle PSB as the rectangle contained by SC, AP, to the rectangle SAP; that is, as twice the rectangle contained by SC, AP, to twice the rectangle SAP; therefore, alternating, the square of SP will be to twice the rectangle contained by SC, AP, as the rectangle PSB to twice the rectangle SAP: but the area TNAK is to the area HPAK as the square of PS to twice the rectangle contained by SC, AP; therefore the area TNAK is to the area HPAK as the rectangle PSB to twice the rectangle SAP; that is, (because the rectangle SAP is equal to the rectangle SPY), as the rectangle PSB to twice the rectangle SPY; that is, as SB to twice PY.

Because the area TNAK is to the rectangle SAK as DB to twice DA; that is, as the rectangle contained by DB, SL, to twice the rectangle contained by DA, SL; and

and the rectangle SAK is to the rectangle DAF as the rectangle SLG to the rectangle DLG; that is, as SL to DL; that is, as twice the rectangle contained by DA, SL, to twice the rectangle ADL; the area TNAK will be to the rectangle DAF as the rectangle contained by DB, SL, to twice the rectangle ADL; and, alternating, the area TNAK will be to the rectangle contained by DB, SL, as the rectangle DAF to twice the rectangle ADL; that is, as AF to twice DL; that is, as the rectangle SAF to twice the rectangle contained by SA, DL; therefore, alternating, the area TNAK will be to the rectangle SAF as the rectangle contained by DB, SL, to twice the rectangle contained by SA, DL: but the rectangle SAF is [II. Tr. 2.] to the area EPAF as SP to PA; that is, (because the rectangle SPY is equal to the rectangle SAP), as SA to PY; that is, as twice the rectangle contained by SA, DL, to twice the rectangle contained by PY, DL; therefore the area TNAK will be to the area EPAF as the rectangle contained by DB, SL, to twice the rectangle contained by PY, DL; and, inversely, the area
EPAF

EPAF will be to the area TNAK as twice the rectangle contained by PY, DL, to the rectangle contained by DB, SL: but the area TNAK is to the area HPAK as SB to twice PY; that is, as the rectangle contained by SB, DL, to twice the rectangle contained by PY, DL; therefore the area EPAF will be to the area HPAK as the rectangle contained by SB, DL, to the rectangle contained by DB, SL. *Q. E. D.*

P R O P. XXIV. *Fig. 82.*

In the straight line PD, let S, D, be two given points; and let the point S be between the points P, D; let EFG, HKG, be two curves, such, that if from any two points A, P, in SP, there be drawn AF, PE, perpendicular to AS, meeting the curve EFG in F, E, and likewise meeting the curve HKG in K, H, the square of SA is to the square of SP as PE to AF, likewise the rectangle DAF is to the rectangle DPE as the rectangle SAK to the rectangle SPH; let the curves EFG, HKG, intersect each other in G; and draw GL perpendicular

dicular to SP, meeting SP in L; let FK be to EH as AS to SP: PL will be to LA in the quadruplicate ratio of PS to SA.

Let PM be to MA in the duplicate ratio of PS to SA: because, from the nature of the curves EFG, HKG, the rectangle DPE is to the rectangle SPH as the rectangle DLG to the rectangle SLG; that is, as DL to LS; that is, as the rectangle contained by DL, SP, to the rectangle LSP; alternating, the rectangle DPE will be to the rectangle contained by DL, SP, as the rectangle SPH to the rectangle LSP; that is, as PH to LS; that is, as the rectangle DPH to the rectangle contained by DP, LS; alternating, the rectangle DPE will be to the rectangle DPH as the rectangle contained by DL, SP, to the rectangle contained by LS, DP; that is, PE will be to PH as the rectangle contained by DL, SP, to the rectangle contained by LS, DP; therefore PE will be to EH as the rectangle contained by DL, SP, to the excess of the rectangle contained by DL, SP, above the rectangle

rectangle contained by LS, DP : but the rectangle contained by DS, PL, is the excess of the rectangle contained by DL, SP, above the rectangle contained by LS, DP ; therefore PE is to EH as the rectangle contained by DL, SP, to the rectangle contained by DS, PL : but the rectangle EPL is to the rectangle contained by EH, PL, as PE to EH ; therefore the rectangle EPL is to the rectangle contained by EH, PL, as the rectangle contained by DL, SP, to the rectangle contained by DS, PL ; and, alternating, the rectangle EPL will be to the rectangle contained by DL, PS, as the rectangle contained by EH, PL, to the rectangle contained by DS, PL ; that is, as EH to DS ; that is, as the rectangle contained by EH, PS, to the rectangle DSP ; and, alternating, the rectangle EPL will be to the rectangle contained by EH, PS, as the rectangle contained by DL, PS, to the rectangle DSP ; that is, as DL to DS. The same way it is shown, that the rectangle FAL is to the rectangle contained by FK, AS, as DL to DS ; therefore the rectangle EPL will be to the rectangle con-

K k tained

tained by EH, PS, as the rectangle FAL to the rectangle contained by FK, AS; and, alternating, the rectangle EPL will be to the rectangle FAL as the rectangle contained by EH, PS, to the rectangle contained by FK, AS: but because EH is to FK as PS to SA, the rectangle contained by EH, PS, will be to the rectangle contained by FK, AS, as the square of SP to the square of SA; that is, as PM to MA; therefore the rectangle EPL will be to the rectangle FAL as PM to MA; that is, as the square of PM to the rectangle PMA; and the rectangle FAL is to the rectangle contained by EP, AL, as AF to EP; that is, as the square of SP to the square of SA; that is, as PM to MA; that is, as the rectangle PMA to the square of MA; therefore the rectangle EPL is to the rectangle contained by EP, AL, as the square of PM to the square of MA; that is, PL is to LA as the square of PM to the square of MA; therefore PL is to LA in the duplicate ratio of PM to MA: but PM is to MA in the duplicate ratio of PS to SA; there
for

fore PL is to LA in the quadruplicate ratio of PS to SA. *Q. E. D.*

P R O P. XXV. *Fig. 83.*

Let there be an ellipse whose transverse axis is AB, and focus S; and in SA take SC equal to SB; draw SD to any point D in the ellipse; and from E, a point in the ellipse indefinitely near to the point D, draw EF perpendicular to SD, meeting SD in F; in SC take SG equal to SD: DF will be to FE in the subduplicate ratio of the rectangle AGC to the rectangle ASB.

Draw SH perpendicular to DH a tangent to the ellipse at the point D; and let SH meet DH in H; let K be the other focus of the ellipse; join KD; and produce SC to L; and let SL be equal to AB.

Because the sum of SD, DK, is equal to AB, that is, equal to SL, and SD is equal to SG, DK will be equal to GL: and because the square of SH is [cor. to prop. 21. Tr. 1.] to the rectangle ASB as SD to DK, the square of SH will be to the rectangle ASB as SG to GL;

K k 2

that

that is, as the square of SG to the rectangle SGL; and, alternating, the square of SH will be to the square of SG as the rectangle ASB to the rectangle SGL; therefore, inverfely, the square of SG will be to the square of SH as the rectangle SGL to the rectangle ASB: but the square of SG is equal to the square of SD; and the rectangle ASB is equal to the rectangle SAL; therefore the square of SD is to the square of SH as the rectangle SGL to the rectangle SAL; therefore, by division, the square of DH will be to the square of HS as the rectangle AGC to the rectangle SAL: but because the point E is indefinitely near to the point D, and the angles EFD, SHD, are equal, because both right, the triangles EFD, SHD, will be equiangular; therefore the square of DF will be to the square of FE as the square of DH to the square of HS; that is, as the rectangle AGC to the rectangle SAL; that is, as the rectangle AGC to the rectangle ASB; therefore DF is to FE in the subduplicate ratio of the rectangle AGC to the rectangle ASB. *Q. E. D.*

P R O P.

P R O P. XXVI. Fig. 84.

Let there be a semicircle AHC , whose diameter is AC ; and let S be a given point in CA produced; let the square of AD be equal to the rectangle ASC ; let EFG be a curve, such, that if from any point K in AC , there be drawn KH perpendicular to AC , meeting the semicircle AHC in H , and the curve EFG in E , the square of SA is to the rectangle HKE as SK to AD , (it is evident that AD drawn perpendicular to SA , at the point A , will be an asymptote to the curve EFG); produce AS to B ; and let SB be equal to SC ; let there be an ellipse described whose transverse axis is AB , and focus S ; if from any point K in AC , there be drawn KE perpendicular to AC , meeting the curve EFG in E ; and with the centre S , and distance SK , a circle be described meeting the ellipse in L ; and SL be joined, meeting the circle described with the centre S , and distance SA , in M : the area contained by AK , KE , the asymptote AD ,
and

and curve EFG indefinitely produced, will be double of the sector ASM.

Let KA be divided into indefinitely little parts in $a, b, c, \&c.$; draw $aF, bG, cN, \&c.$ perpendicular to KA, meeting the curve EFG in F, G, N, $\&c.$; with the centre S, and distances $Sa, Sb, Sc, \&c.$ let there be circles described, meeting the ellipse in $d, e, f, \&c.$; join $Sd, Se, Sf, \&c.$ meeting the circle described; with the centre S, and distance SA, in $g, h, k, \&c.$ draw dl perpendicular to SL, meeting SL in l .

Because the square of KH is equal to the rectangle AKC, and the square of AD equal to the rectangle ASC; that is, equal to the rectangle ASB, because SB, SC, are equal; therefore HK will be to AD in the subduplicate ratio of the rectangle AKC to the rectangle ASB: but Ll is [25.] to ld in the subduplicate ratio of the rectangle AKC to the rectangle ASB; therefore Ll is to ld as KH to AD; that is, as the rectangle SKH to the rectangle contained by AD, SK: but ld is to Mg as SL to SM; that is, as SK to SA; that is, as the rectangle

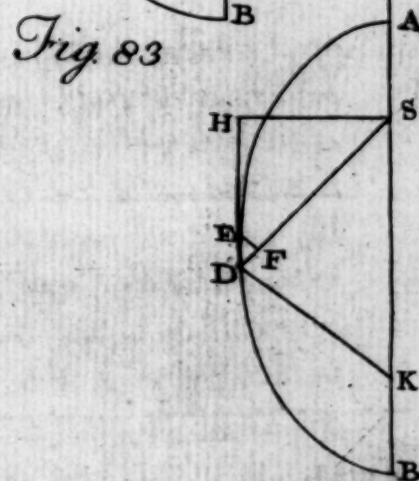
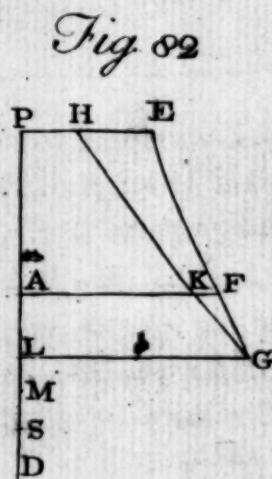
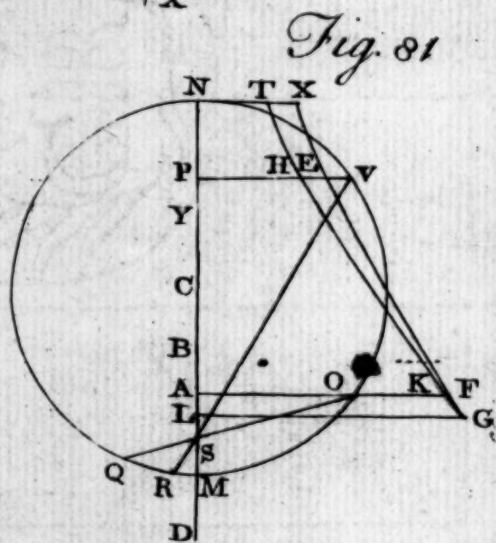
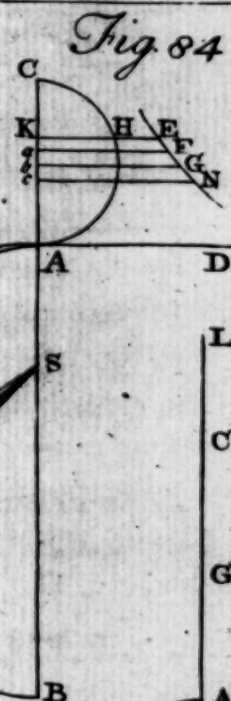
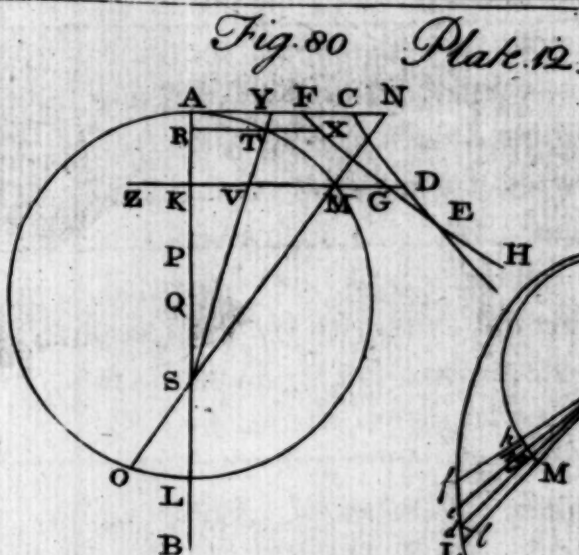
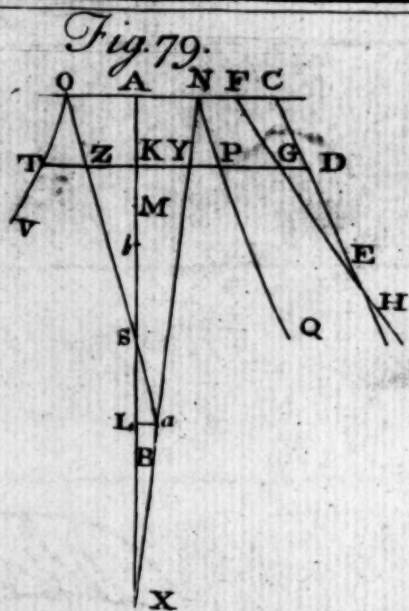
angle

angle contained by AD, SK, to the rectangle DAS; therefore Ll is to Mg as the rectangle SKH to the rectangle DAS. Again, because, the square of AS is to the rectangle HKE as SK to AD; that is, as the rectangle ASK to the rectangle DAS; alternating, the square of AS will be to the rectangle ASK as the rectangle HKE to the rectangle DAS: but the square of AS is to the rectangle ASK as AS to SK; that is, as the rectangle contained by AS, KH, to the rectangle SKH; therefore the rectangle contained by AS, KH, will be to the rectangle SKH as the rectangle HKE to the rectangle DAS; therefore, alternating, the rectangle contained by AS, KH, will be to the rectangle HKE as the rectangle SKH to the rectangle DAS: but AS is to KE as the rectangle contained by AS, KH, to the rectangle HKE; and Ll is to Mg as the rectangle SKH to the rectangle DAS; therefore AS is to KE as Ll to Mg : but because SK is equal to SL, and Sa equal to Sd , or Sl , (because the point d is indefinitely near to the point L); therefore Ll is equal to Ka ; therefore AS is to KE as Ka to Mg ; therefore

therefore the rectangle EKa is equal to the rectangle contained by SA , Mg ; that is, equal to the rectangle SMg : but the rectangle EKa is equal to the space $EKaF$; and the rectangle SMg is double of the sector SMg ; therefore the space $EKaF$ is double of the sector SMg . The same way it is shown, that the space $FabG$ is double of the sector Sgb ; and the space gbN double of the sector Sbk , and so on; therefore it is evident, that the space contained by EK , KA , the asymptote AD , and the curve EFG indefinitely produced, is double of the sector ASM , $\mathcal{Q}$, *E. D.*

P R O P. XXVII. *Fig. 85.*

Suppose a body acted upon by a centripetal force tending to the centre S, that is reciprocally as the square of the distance from the centre, to descend in the straight line AS from A to B; and at the point B let the body be projected in the direction BD, perpendicular to SB, with the velocity that would be acquired in falling from A to B; produce BS to F; and let BF be equal to AS: the body will describe



describe an ellipse whose transverse axis is FB, and focus S.

In AS take SN equal to AB; and upon AS, BN, describe the semicircles ALS, BPN; suppose the body at any point H in its orbit; join SH; with the centre S, and distance SH, describe a circle meeting AS in K; draw BG, KL, perpendicular to AS, meeting the semicircle ALS in G, L; and join SL, meeting BG in M; let KL meet the semicircle BPN in P; and with the centre S, and distance SB, describe a circle meeting SH in O; let SB be to SQ as the velocity that would be acquired in falling from A to K to the velocity that would be acquired in falling from A to B; that is, [12. Tr. 2.] as BM to BG; and let the excess of the square of SK above the square of SQ be to the square of SQ as the square of SK to the square of SR; let TVX be a curve, such, that drawing KV perpendicular to SK, meeting the curve TVX in X, SR will be to KV as the square of SK to the square of SB; therefore [19. Tr. 1.] the area contained by BK,

L I

KV,

KV, the curve VT, and its asymptote BG indefinitely produced, is double of the sector BSO.

Because SK is to SB as KL to BM, and SB is to SQ as BM to BG, SK will be to SQ as KL to BG; therefore the square of SK will be to the square of SQ as the square of KL to the square of BG; therefore, by division, the excess of the square of SK above the square of SQ will be to the square of SQ as the excess of the square of KL above the square of BG to the square of BG; that is, as the rectangle BKN to the square of BG; that is, as the square of KP to the square of BG: but the excess of the square of SK above the square of SQ is to the square of SQ as the square of SK to the square of SR; therefore the square of SK is to the square of SR as the square of KP to the square of BG; therefore SK is to SR as KP to BG: but because the rectangle contained by SK, SR, is to the rectangle SKV as SR to KV; that is, as the square of SK to the square of SB; the rectangle contained by SK, SR, will be to the square of SK as the rectangle SKV to the square of SB;

SB; therefore SR will be to SK as the rectangle SKV to the square of SB; therefore, inversely, SK will be to SR as the square of SB to the rectangle SKV: but SK is to SR as KP to BG; therefore the square of SB will be to the rectangle SKV as KP to BG: but the rectangle SKV is to the rectangle PKV as SK to KP; therefore the square of SB will be to the rectangle PKV as SK to BG: and because the area contained by BK, KV, the curve VT, and its asymptote BG, indefinitely produced, is double of the sector BSO; therefore it is evident, [26.] that the point H is in the ellipse whose transverse axis is BF, and focus S.

Q. E. D.

P R O P. XXVIII. Fig. 86.

In the straight line AS, let S, B, be two given points; and let the point S be between the points A, B; let CDE, FGH, be two curves, such, that if from any two points A, K, in AS, there be drawn AC, KD, perpendicular to AS, meeting the curve CDE in C, D, and likewise meeting the

curve FGH in F, G, the square of SK is to the square of SA as AC to KD, likewise the rectangle BAC is to the rectangle BKD as the rectangle SAF to the rectangle SKG; suppose a body acted upon by a centripetal force tending to the centre S to descend in the straight line AS; and the centripetal force at any two points A, K, to be as AF to KG; let the body at the point L be projected in the direction LM, perpendicular to AS; with the velocity that would be acquired in falling from A to L, it is required to determine the path of the body.

In BS take the point N, such, that BN is to NS as BA to AS; and in NA take NP equal to AL; upon AN, LP, describe the semicircles AON, PVL; suppose the body at any point Q, in its orbit; join SQ, with the centre S, and distance SQ, describe a circle, meeting AS in R; draw LO, RT, perpendicular to AS, meeting the semicircle AON in O, T; join ST, meeting LO in X; and let RT meet the

the semicircle PVL in V; with the centre S, and distance SL, describe a circle meeting SQ in V; let SL be to Sd as the velocity that would be acquired by falling from A to R. to the velocity that would be acquired by falling from A to L; that is, [cor. 1. prop. 22.] as LX to LO; and let the excess of the square of SR above the square of Sd be to the square of Sd as the square of SR to the square of Sf ; let abc be a curve, such, that drawing, from any point R in AL, Rb perpendicular to AL, meeting the curve abc in b , Sf will be to Rb as the square of SR to the square of SL. It is evident, that LO is an asymptote to the curve abc ; therefore [19. Tr. 1.] the area contained by LR, Rb, the curve ba , and its asymptote LO indefinitely produced, will be double of the sector LSV; produce LS to g ; and let Sg be equal to SP; and with the transverse axis Lg, and focus S, let an ellipse be described; and let the circle described with the centre S, and distance SQ, meet the ellipse in Y; and join SY, meeting the circle LV in Z; let the square of Lk be equal to the rectangle LSg; that is,

equal

equal to the rectangle LSP ; let pqr be a curve, such, that if from any point R in PL there be drawn Rb , perpendicular to PL , meeting the curves abc , pqr , in b , q , Rb will be to Rq as LO to Lk . It is evident, that LO will be an asymptote to the curve pqr ; and that the area contained by Rb , RL , the curve ba , and the asymptote LO , indefinitely produced, will be to the area contained by qR , RL , the curve qp , and the asymptote LO indefinitely produced, as LO to Lk .

Because SR is to SL as RT to LX ; and SL is to Sd as LX to LO ; SR will be to Sd , as RT to LO ; therefore the square of SR will be to the square of Sd as the square of RT to the square of LO ; therefore the excess of the square of SR above the square of Sd will be to the square of Sd as the excess of the square of RT above the square of LO to the square of LO ; that is, as the rectangle PRL to the square of LO ; that is, as the square of RV to the square of LO : but the excess of the square of SR above the square of Sd is to the square of Sd as the square of SR to the square of Sf ; therefore the

the square of RV is to the square of LO as the square of SR to the square of Sf ; therefore RV is to LO as SR to Sf ; therefore, alternating, RV will be to SR as LO to Sf ; that is, LO will be to Sf as RV to SR ; that is, as the rectangle SRV to the square of SR : but Sf is to Rb as the square of SR to the square of SL ; therefore LO will be to Rb as the rectangle SRV to the square of SL ; that is, the square of SL will be to the rectangle SRV as Rb to LO : but the rectangle SRV is to the rectangle VRb as SR to Rb ; therefore the square of SL is to the rectangle VRb as SR to LO : but the rectangle VRb is to the rectangle VRq as Rb to Rq ; that is, as LO to Lk ; therefore the square of SL is to the rectangle VRq as SR to Lk ; therefore [26.] the area contained by Rq , RL , the curve qp , and the asymptote LO indefinitely produced, is double of the sector LSZ : but the area contained by Rb , RL , the curve ba , and the asymptote LO indefinitely produced, is double of the sector LSV ; therefore the sector LSV will be to the sector LSZ as the area contained by Rb , RL , the curve ba ,
and

and the asymptote LO indefinitely produced, to the area contained by qR , RL , the curve qp , and the asymptote LO indefinitely produced; that is, as LO to Lk ; therefore the angle LSQ is to the angle LSY as LO to Lk : but because the square of LO is equal to the rectangle ALN ; that is, equal to the rectangle PNL , and the square of Lk is equal to the rectangle PSL , LO will be to Lk in the subduplicate ratio of the rectangle PNL to the rectangle PSL ; therefore the angle LSQ is to the angle LSY in the subduplicate ratio of the rectangle PNL to the rectangle PSL . Q. E. I.

From whence may be deduced the following

CONSTRUCTION.

In BS take the point N , such, that BN is to NS as BA to AS ; and in NA take NP equal to AL ; produce LS to g ; and let Sg be equal to SP ; and with the transverse axis Lg , and focus S , let there be an ellipse described; let Lmn be a curve, so related to the ellipse, that if, with the centre S ,
and

and any distance SY, a circle be described, meeting the ellipse in Y, and the curve Lmn in Q, and SY, SQ, be drawn, the angle LSQ is to the angle LSY in the subduplicate ratio of the rectangle PNL to the rectangle PSL; the curve Lmn will be the path of the body.

The demonstration of this construction is obvious from the proposition.

COR. It is evident, that the line of the apsides of this trajectory has a progressive motion, and that the angle described by the body from any apsis to its return to the same apsis again is to four right angles in the subduplicate ratio of the rectangle PNL to the rectangle PSL; likewise the area described by the body revolving from any apsis to the same apsis again will be to the area of the ellipse whose transverse axis is Lg, and focus S, in the subduplicate ratio of the rectangle PNL to the rectangle PSL.

M m P R O P.

PROP. XXIX. Fig. 87.

Let S, D , be two given points in the straight line AC , and let the point S be between the points C, D ; let EFG, HKG , be two curves, such, that if from any two points A, C , in CS , there be drawn AF, CE , perpendicular to CS , meeting the curve EFG in F, E , and likewise meeting the curve HKG in K, H , the square of SA is to the square of SC as CE to AF , likewise the rectangle DAF is to the rectangle DCE as the rectangle SAK to the rectangle SCH ; let the curves EFG, HKG , intersect each other in the point G , and draw GL perpendicular to AS , meeting AS in L ; let the point A be between the points C, L ; in DS take the point M , such, that DM is to MS as DC to CS ; in MC take MN equal to CA ; produce AS to B , and let SB be equal to SN ; with the transverse axis AB , and focus S , let there be an ellipse described; let Aab be a curve, so related to the ellipse, that if, with the centre S , and any distance Sb , a circle be described,

described, meeting the curve Aab in b , and the ellipse in c , and Sb , Sc , be drawn, the angle ASb is to the angle ASc in the subduplicate ratio of the rectangle AMN to the rectangle ASN ; suppose a body acted upon by a centripetal force tending to the centre S to descend in the straight line CS , and the centripetal force at any two points A , C , to be as AK to CH ; let the body at the point A be projected in the direction AO , perpendicular to AS , with the velocity that would be acquired by falling from C to A ; the body will, by the last proposition, describe the curve Aab . Again, in AC take AP equal to SB ; and suppose a body acted upon by a centripetal force tending to the centre S , that is reciprocally as the square of the distance from the centre S , to descend in the straight line PS ; and let this centripetal force at the point A be to the other centripetal force at the point A as AF to AK ; let the body at the point A be projected in the direction AO perpendicular to AS , with the velocity that would be acquired by falling from P to A ; the body will [27.] describe the ellipse whose transverse axis is AB ,

and focus S: the time the body describing the curve Aab takes to revolve from any apsis to the same apsis again will be to the periodic time of the body describing the ellipse in the subduplicate ratio of DL to LS.

Upon CM let there be a semicircle described, meeting AK in Q; join SQ meeting the circle CQM in R; bisect MC in T; and draw PV perpendicular to SP, meeting the curve EFG in V; let Ad be a part of the curve Aab described in an indefinitely little part of time; and let Af be a part of the ellipse described in the same time that Ad was described; join Sd, Sf; in TA take TX, such, that the rectangle STX is equal to the square of TA; let the area of the ellipse whose transverse axis is AB, and focus S, be to the space Z as the area ASf to the area ASd.

Because Ad, Af, are described in the same time, the area ASd will be to the area ASf as the velocity at the point A of the body describing the curve Aab to the velocity at the point A of the body describing the ellipse: but the velocity of the body at the

the point A describing the curve Aab is to the velocity of the body at the point A describing the ellipse in the subduplicate ratio of the area $HCAK$ to the area $VPAF$; therefore the area ASd is to the area ASf in the subduplicate ratio of the area $HCAK$ to the area $VPAF$. Because DM is to MS as DC to CS , and MC is bisected in T ; therefore [26. Tr. 2.] the rectangle DTS is equal to the square of TM : but the rectangle STX is equal to the square of TA ; therefore the rectangle contained by DX , TS , is equal to the rectangle MAC ; that is, equal to the square of AQ ; and [5. Tr. 2.] the rectangle SQR is equal to twice the rectangle contained by DA , TS ; therefore the square of AQ will be to the rectangle SQR as the rectangle contained by DX , TS , to twice the rectangle contained by DA , TS ; that is, as DX to twice DA : but [22.] the area $HCAK$ is to the rectangle SAK as the square of AQ to the rectangle SQR ; therefore the area $HCAK$ is to the rectangle SAK as DX to twice DA ; that is, as the rectangle contained by DX , AF , to twice the rectangle DAF ; and, alternating,

alternating, the area HCAK is to the rectangle contained by DX, AF, as the rectangle SAK to twice the rectangle DAF: but because the rectangle SAK is to the rectangle DAF as SL to LD, the rectangle SAK will be to twice the rectangle DAF as SL to twice LD; therefore the area HCAK will be to the rectangle contained by DX, AF, as SL to twice LD; that is, as the rectangle contained by SL, DX, to twice the rectangle LDX; and, alternating, the area HCAK will be to the rectangle contained by SL, DX, as the rectangle contained by DX, AF, to twice the rectangle LDX; that is, as AF to twice LD; that is, as the rectangle SAF to twice the rectangle contained by AS, LD; therefore, alternating, the area HCAK will be to the rectangle SAF as the rectangle contained by SL, DX, to twice the rectangle contained by AS, LD.

Because MN is equal to AC, therefore MA, CN, are equal; therefore TA is equal to TN: and because the rectangle STX is equal to the square of TA; therefore the rectangle TSX will be equal to the rectangle

rectangle ASN ; therefore AS will be to SX as TS to SN ; therefore twice AS will be to SX as twice TS to SN : but because AP is equal to SB , that is, equal to SN , SP will be equal to twice ST ; therefore twice AS will be to SX as SP to PA : but the rectangle SAF is to the area $VPAF$ [II. Tr. 2.] as SP to PA ; therefore the rectangle SAF will be to the area $VPAF$ as twice AS to SX ; that is, as twice the rectangle contained by AS , LD , to the rectangle contained by SX , LD : but because the area $HCAK$ is to the rectangle SAF as the rectangle contained by SL , DX , to twice the rectangle contained by AS , LD , and the rectangle SAF is to the area $VPAF$ as twice the rectangle contained by AS , LD , to the rectangle contained by SX , LD ; therefore the area $HCAK$ is to the area $VPAF$ as the rectangle contained by SL , DX , to the rectangle contained by SX , LD : and because the area ASd is to the area ASf in the subduplicate ratio of the area $HCAK$ to the area $VPAF$; therefore the area ASd will be to the area ASf in the subduplicate ratio of the

the

the rectangle contained by SL , DX , to the rectangle contained by SX , LD ; and, inversely, the area ASf will be to the area ASd in the subduplicate ratio of the rectangle contained by SX , DL , to the rectangle contained by SL , DX . Because the area described by the body revolving in the curve Aab from any apsis to the same apsis again is [cor. prop. 28.] to the area of the ellipse whose transverse axis is AB , and focus S , in the subduplicate ratio of the rectangle AMN to the rectangle ASN ; and the rectangle AMN is equal to the rectangle contained by DX , TS ; and the rectangle ASN equal to the rectangle TSX ; therefore the area described by the body revolving in the curve Aab from any apsis to the same apsis again is to the area of the ellipse whose transverse axis is AB , and focus S , in the subduplicate ratio of the rectangle contained by DX , TS , to the rectangle TSX ; that is, in the subduplicate ratio of DX to SX ; that is, in the subduplicate ratio of the rectangle LDX to the rectangle contained by SX , LD . Because the area ASf is to the

the area ASd as the area of the ellipse whose transverse axis is AB , and focus S , to the space Z , the area ASf will be to the area of the ellipse whose transverse axis is AB , and focus S , as the area ASd to the space Z .

Because the time the body describing the curve Aab takes to revolve from any apsis to the same apsis again is to the time in which the part Ad is described as the area described by the body revolving from any apsis to the same apsis again to the area ASd , and the time in which the part Ad is described is equal to the time in which the part Af of the ellipse is described; therefore the time the body describing the curve Aab takes to revolve from any apsis to the same apsis again is to the time in which the part Af of the ellipse is described, as the area described by the body revolving from any apsis to the same apsis again is to the area ASd ; but the time in which the part Af is described is to the periodic time of the body describing the ellipse whose transverse axis is AB , and focus S , as the area ASf to the area of the ellipse; that is, as the area ASd to the space Z ; therefore, by equality,

Nn the

the time the body describing the curve *Aab* takes to revolve from any apsis to the same apsis again is to the periodic time of the body describing the ellipse, as the area described by the body revolving from any apsis to the same apsis again to the space *Z*. Because the area described by the body revolving in the curve *Aab* from any apsis to the same apsis again is to the area of the ellipse whose transverse axis is *AB*, and focus *S*, in the subduplicate ratio of the rectangle *LDX* to the rectangle contained by *SX*, *LD*, and the area of the ellipse whose transverse axis is *AB*, and focus *S*, is to the space *Z* as the area *ASf* to the area *ASd*; that is, in the subduplicate ratio of the rectangle contained by *SX*, *LD*, to the rectangle contained by *SL*, *DX*; therefore, by equality, the area described by the body revolving in the curve *Aab* from any apsis to the same apsis again is to the space *Z* in the subduplicate ratio of the rectangle *LDX* to the rectangle contained by *SL*, *DX*; that is, in the subduplicate ratio of *DL* to *LS*: but the time the body describing the curve *Aab* takes to revolve from any apsis to the same apsis again is to the periodic time of the body

dy describing the ellipse, as the area described by the body revolving from any apsis to the same apsis again is to the space Z ; therefore the time the body describing the curve Aab takes to revolve from any apsis to the same apsis again is to the periodic time of the body describing the ellipse in the subduplicate ratio of DL to LS . Q. E. D.

COR. I. The angle described by the body revolving in the curve Aab from any apsis to its return to the same apsis again will be to four right angles in the subduplicate ratio of DX to XS .

Because the rectangle AMN is equal to the excess of the square of TM above the square of TA , and the rectangle DTS is equal to the square of TM , and the rectangle STX equal to the square of TA ; therefore the rectangle AMN will be equal to the rectangle contained by DX , TS ; and because the rectangle ASN is equal to the rectangle TSX ; therefore the rectangle AMN will be to the rectangle ASN as the rectangle contained by DX , TS , to the rectangle TSX ; that is, as DX to XS : but

$N n 2$

the

the angle ASb is to the angle ASc in the subduplicate ratio of the rectangle AMN to the rectangle ASN ; therefore the angle ASb will be to the angle ASc in the subduplicate ratio of DX to XS : but it is evident, that the angle described by the body revolving in the curve Aab from any apsis to its return to the same apsis again is to four right angles as the angle ASb to the angle ASc ; therefore the angle described by the body revolving in the curve Aab from any apsis to its return to the same apsis is to four right angles in the subduplicate ratio of DX to XS . $\mathcal{Q}$. E . D .

COR. 2. If the centrifugal force at the point A arising from the projectile force in the direction AO perpendicular to AS , be equal to the centripetal force AF ; the rectangle contained by DL , AS , will be equal to the rectangle contained by DX , SL .

Because the centrifugal force at the point A arising from the projectile force in the direction AO , perpendicular to AS , is [cor. 2. to prop. 22.] to the centripetal force AK as twice the square of AQ to the rectangle SQR ;

SQR ; that is, as twice the rectangle contained by DX , TS , to twice the rectangle contained by DA , TS ; that is, as DX to DA ; AF will be to AK as DX to DA ; therefore the rectangle DAF will be equal to the rectangle contained by DX , AK ; but because DL is to LS as the rectangle DAF to the rectangle SAK ; therefore DL will be to LS as the rectangle contained by DX , AK , to the rectangle SAK ; that is, as DX to AS ; therefore the rectangle contained by DL , AS , will be equal to the rectangle contained by DX , SL . Q. E. D.

P R O P. XXX. Fig. 88.

Let B , A , D , be three points in a straight line, and let the point A be between the points B , D ; let E be a point in AD , such, that BE is to EA in the quadruplicate ratio of BD to DA ; bisect AB in C ; if BA be little in comparison of BD , DC will be nearly quadruple of CE .

In AD take the point F , such, that BF is to FA in the duplicate ratio of BD to DA ;

DA; in BC take CK, CL, such, that the rectangle DCK is equal to the square of CA; and likewise the rectangle FCL equal to the square of AC.

Because BE is to EA in the quadruplicate ratio of BD to DA, and BF is to FA in the duplicate ratio of BD to DA, the square of BD will be to the square of DA as BF to FA, and likewise the square of BF will be to the square of FA as BE to EA. Again, because the square of BD is to the square of DA as BF to FA, the sum of the squares of BD, DA, will be to the difference of the squares of BD, DA, as the sum of BF, FA, to AB; that is, as FC to CA; but because the sum of the squares of BD, DA, is double of the sum of the squares of CD, CA; and the difference of the squares of BD, DA, double the rectangle contained by DC, AB; therefore the sum of the squares of DC, CA, will be to the rectangle contained by DC, AB, as FC to CA: but because the square of CA is equal to the rectangle DCK, the sum of the squares of DC, CA, will be equal to the rectangle CDK; therefore the sum of the squares of DC, CA, will

will be to the rectangle contained by DC, AB, as the rectangle CDK to the rectangle contained by CD, AB; that is, as DK to AB; therefore DK will be to AB as FC to CA: but AB is double of CA, therefore DK is double of FC. The same way it is shown, that FL is double of EC: and because the rectangles DCK, FCL, are each of them equal to the square of CA, and AC is little in comparison of CD; therefore CK, CL, will be little in comparison of DC, CF; therefore DK, DC, will be nearly equal; and likewise FL, FC, nearly equal; therefore DC will be nearly double of CF, and CF nearly double of CE; therefore DC will be nearly quadruple of EC. 2. E. D.

TRACT

TRACT IV,

PROP. I. Fig. 89.

LET there be an ellipse whose transverse axis is AB , and centre C ; let CD be half the lesser axis; in CA take CE , CG , equal to one another; and take CF in CA , such, that the square of CE is to the rectangle EFG as the square of CA to the square of CD ; with the centre C , and distance CF , let there be a circle FKP described; from C draw CH to any point H in the ellipse meeting the circle FKP in K ; and draw KL perpendicular to AB , meeting AB in L ; the square of CE will be to the rectangle ELG as the square of CH to the square of CD .

With the centre C , and distance CD , describe the circle DQO ; draw HR perpendicular to CD , meeting CD in R , and the circle DQO in Q , S ; and draw KM perpendicular to CD , meeting CD in M ; and

and let CH meet the circle DQO in N, O;
let CD meet the ellipse in T.

Because the square of CE is to the rectangle EFG as the square of CA to the square of CD; that is, as the square of HR to the rectangle DRT, or the square of QR; therefore, conversely, the square of CE will be to the square of CF as the square of HR to the rectangle QHS; and the square of CF, that is, the square of CK is to the square of KM as the square of CH to the square of HR; therefore, by perturbate equality, the square of CE is to the square of KM as the square of CH to the rectangle QHS: but the square of KM is equal to the square of CL; and the rectangle QHS is equal to the rectangle NHO; therefore the square of CE is to the square of CL as the square of CH to the rectangle NHO; and therefore, conversely, the square of CE is to the rectangle ELG as the square of CH to the square of CN, or CD.

Q. E. D.

Q o

PROP,

PROP. II. Fig. 90.

Let there be a circle whose diameter is AB, and centre C, and let DE be the diameter perpendicular to AB; in CA take any point F, and in CB take CG equal to CF; in CA produced take any point H, and in CB produced take CK equal to CH; in CA take CL such, that the square of CL is to the square of CA as the rectangle HFK to the rectangle HAK; and in CD take CM such, that the square of CD is to the square CM as the square of CH to the rectangle HFK; with the semiaxes CL, CM, let there be an ellipse LMNO described; draw CP to any point P in the ellipse, meeting the circle ADB in Q, R; from Q draw QT perpendicular to AB, meeting AB in T: the square of CP will be to the rectangle QPR as the rectangle HFK to the rectangle FTG; likewise the square of CQ will be to the rectangle QPR as the rectangle HTK to the rectangle FTG.

Draw

Draw FS perpendicular to AB, meeting the circle ADB in S.

Because the square of CL is to the square of CA as the rectangle HFK to the rectangle HAK; by division, the rectangle LAN will be to the square of CA as the rectangle FAG; that is, the rectangle AFB to the rectangle HAK; and, alternating, the rectangle LAN will be to the rectangle AFB as the square of CA to the rectangle HAK; therefore, by composition, the rectangle LFN will be to the rectangle AFB, that is, to the square of FS, as the square of CH to the rectangle HAK: but because the square of CL is to the square of CA as the rectangle HFK to the rectangle HAK; and the square of CA, or CD, is to the square of CM as the square of CH to the rectangle HFK, the square of CL will be to the square of CM as the square of CH to the rectangle HAK; therefore the rectangle LFN will be to the square of FS as the square of CL to the square of CM; therefore the point S is in the ellipse: and because the square of CL

is to the square of CA as the rectangle HFK to the rectangle HAK, and the rectangle HFK is greater than the rectangle HAK, the square of CL is greater than the square of CA; therefore CL is greater than CA; therefore the point L is without the circle; and therefore the portion of the ellipse intercepted between L, S, falls without the circle. Again, because the square of CD is to the square of CM as the square of CH to the rectangle HFK, and the square of CH is greater than the rectangle HFK; therefore the square of CD is greater than the square of CM; therefore CD is greater than CM; therefore the point M falls within the circle; and therefore the portion of the ellipse intercepted between M, S, falls within the circle.

Again, because the square of CH is to the rectangle HFK as the square of CD, or CA, to the square of CM; inversely, the rectangle HFK will be to the square of CH as the square of CM to the square of CA; and the square of CH is [1.] to the rectangle HFK as the square of CP to the square

square of CM; therefore, by perturbed equality, the rectangle HFK will be to the rectangle HTK as the square of CP to the square of CA, or CQ; therefore, conversely, the rectangle HFK will be to the rectangle FTG as the square of CP to the rectangle QPR; and, by division, the rectangle HTK will be to the rectangle FTG as the square of CQ to the rectangle QPR. *Q. E. D.*

P R O P. III. *Fig. 90.*

The same construction remaining as in the former proposition, if the rectangle AFB be not less than the square of CF, the rectangle LCM will be greater than the square of CA.

Let the square of the line p be equal to the rectangle LAN, and the square of the line q equal to the rectangle HFK.

Because the rectangle AFB, that is, the square of FS, is not less than the square of CF, therefore FS is not less than CF: and because CL is greater than CA, therefore the rectangle contained by CL, FS, is greater

greater than the rectangle contained by CA, CF: and because the rectangle HFK, that is, the square of q is to the rectangle FAG as the square of CL to the rectangle LAN; and the rectangle FAG, that is, the rectangle AFB, is equal to the square of FS, and the rectangle LAN equal to the square of p ; the square of q will be to the square of FS as the square of CL to the square of p ; therefore q will be to FS as CL to p ; therefore the rectangle contained by p and q is equal to the rectangle contained by CL, FS: but the rectangle contained by CL, FS, is greater than the rectangle contained by CA, CF; therefore the rectangle contained by p and q is greater than the rectangle contained by CA, CF; therefore the ratio of p to CA is greater than the ratio of CF to q ; therefore the ratio of the square of p to the square of CA is greater than the ratio of the square of CF to the square of q ; that is, the ratio of the rectangle LAN to the square of CA is greater than the ratio of the square of CF to the rectangle HFK: but [2.] the rectangle DME is to the square of CM as the square of CF to the rectangle HFK;

HFK; therefore the ratio of the rectangle LAN to the square of CA is greater than the ratio of the rectangle DME to the square of CM; therefore, by composition, the ratio of the square of CL to the square of CA is greater than the ratio of the square of CD, or CA, to the square of CM; therefore the ratio of CL to CA is greater than the ratio of CA to CM; and therefore the rectangle LCM is greater than the square of CA.
Q. E. D.

PROP. IV. Fig. 91.

Let A, B, be two points in the semidiameter of a circle whose centre is C, and let the rectangle ACB be equal to the square of the semidiameter; bisect AB in D; from the point A draw AF to any point F in the circle; and draw FM perpendicular to AB, meeting AB in M; the square of AF will be equal to twice the rectangle contained by AC, DM.

Draw DE perpendicular to AB; and draw FE parallel to AB, meeting DE in E,

E, and the circle in H; join CE, CF, CH, and let CE meet the circle in K, L; produce AC to G, and let AC, CG, be equal; join GF, GH, AH, AE.

Because the square of CD is equal to the rectangle ACB together with the square of AD; that is, equal to the square of the semidiameter together with the square of AD; therefore, adding the square of DE to both, the square of CE will be equal to the square of the semidiameter together with the square of AE; therefore, taking the square of the semidiameter from both, the square of AE will be equal to the rectangle KEL; that is, equal to the rectangle FEH; and therefore FE is to AE as AE to EH; therefore the triangles AEF, AHE, are similar; and the angle EAF will be equal to the angle AHE; that is, equal to the angle HAG. Again, because the angle ACF is equal to the angle CFH; that is, equal to the angle CHF, or GCH; the angle ACH will be equal to the angle GCF: and because AC, CH, are equal to GC, CF, the triangles ACH, GCF, will be every way equal; and the angle CGF will be equal to the
angle

Plak. 13.

Fig. 85.

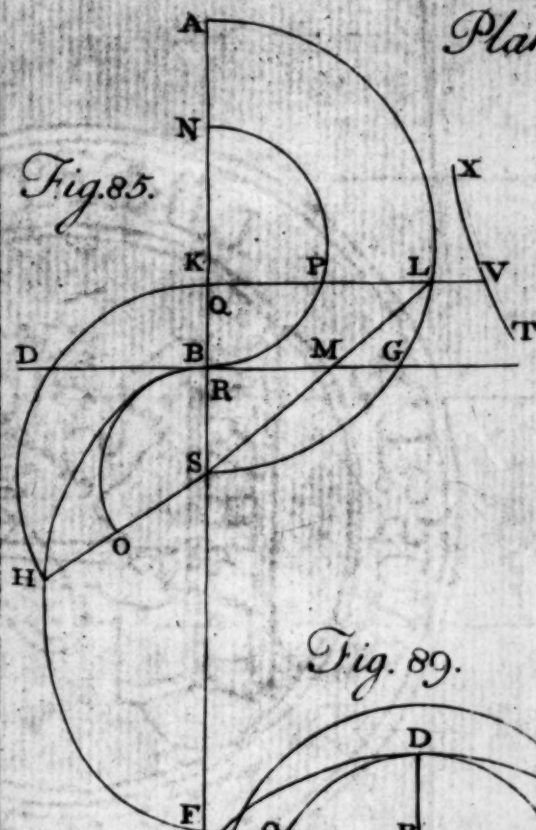


Fig. 86.

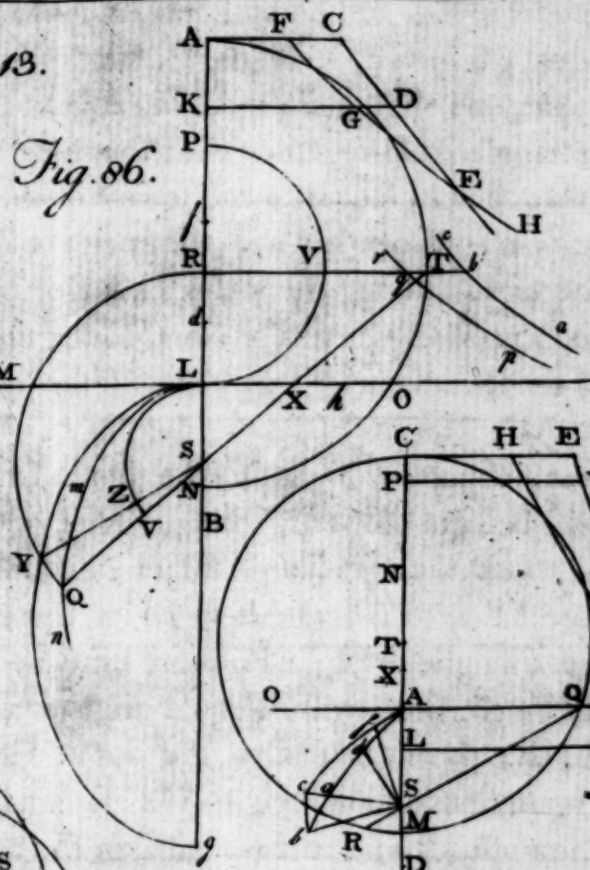


Fig. 89.

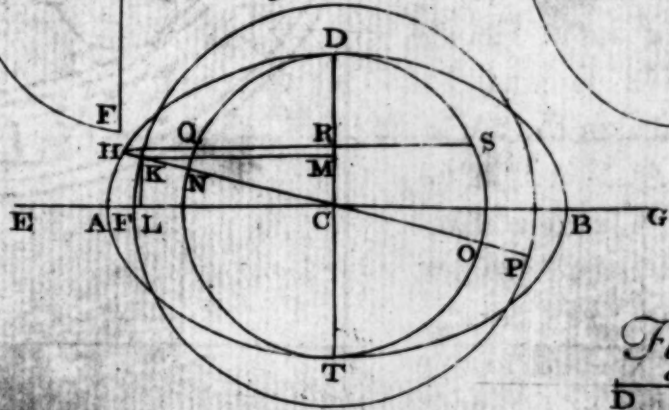


Fig. 88.

D F E A C K L B

angle HAG; that is, equal to the angle EAF; and because the angles EFA, FAG, are equal, the triangles AEF, FAG, will be similar; and therefore EF will be to FA as FA to AG; therefore the square of AF is equal to the rectangle contained by EF, AG; that is, equal to the rectangle contained by DM, AG; that is, equal to twice the rectangle contained by AC, DM.

Q. E. D.

PROP. V. Fig. 92.

Let ABC, DEF, be two circles described by bodies revolving with an equable motion, and therefore acted upon by centripetal forces tending to the centres G, H; let GA be to HK in the duplicate ratio of the periodic time in which the circle ABC is described to the periodic time in which the circle DEF is described: the centripetal force of the body in the circumference ABC tending to the centre G is to the centripetal force of the body in the circumference of the circle DEF tending to the centre H as KH to HD.

P P

Let

Let the arcs AB, DE, be described in an indefinitely small part of time; and let the circumference ABC be to the arc AN as the circumference DEF to the arc DE; let GA, DH, meet the circles in C, F; and draw BL, EM, perpendicular to AC, DF, meeting AC, DF, in L, M.

Because the motions are equable, the time in which the circumference ABC is described will be to the time in which the arcs AB or DE is described as the circumference ABC to the arc AB; and the time in which the arc DE is described is to the time in which the circumference DEF is described as the arc DE to the circumference DEF; that is, as the arc AN to the circumference ABC; therefore, by equality, the time in which the circumference ABC is described is to the time in which the circumference DEF is described as the arc AN to the arc AB; therefore the square of the arc AN will be to the square of the arc AB in the duplicate ratio of the periodic time in which the circle ABC is described to the periodic time in which the circle DEF

DEF is described; therefore GA will be to HK as the square of the arc AN to the square of the arc AB: but GA is to HK as the square of GA to the rectangle contained by GA, HK; therefore the square of GA is to the rectangle contained by GA, HK, as the square of the arc AN to the square of the arc AB; and therefore the square of the arc AN will be to the square of the arc AB as the square of GA to the rectangle contained by GA, HK. Again, because the arc DE is to the circumference DEF as the arc AN to the circumference ABC, the arc DE will be to the arc AN as the circumference DEF to the circumference ABC; that is, as HD to GA; therefore the square of the arc DE will be to the square of the arc AN as the square of HD to the square of GA: but the square of the arc AN is to the square of the arc AB as the square of GA to the rectangle contained by GA, HK; therefore the square of the arc DE will be to the square of the arc AB as the square of HD to the rectangle contained by HK, GA: but because the arcs DE, AB, are indefinitely lit-

the, the square of the arc DE will be to the square of the arc AB as the rectangle FDM to the rectangle CAL; therefore the square of HD will be to the rectangle contained by GA, HK, as the rectangle FDM to the rectangle CAL; that is, as the rectangle HDM to the rectangle GAL; and, alternating, the square of HD will be to the rectangle HDM as the rectangle contained by GA, HK, to the rectangle GAL; therefore HD will be to DM as HK to AL; and, alternating, DH will be to HK as DM to AL; but because the arcs AB, DE, are described in an indefinitely little time, DM will be to AL as the centripetal force at D to the centripetal force at A; therefore the centripetal force at D is to the centripetal force at A as HD to HK; and, inversely, the centripetal force at A will be to the centripetal force at D as HK to HD.

Q. E. D.

COR. If the circles ABC, DEF, be equal, the centripetal forces will be to each other reciprocally in the duplicate ratio of the periodic times.

Because

Because the circles ABC, DEF, are equal, HD, AG, will be equal; therefore HK will be to HD as HK to AG; but because AG is to HK in the duplicate ratio of the periodic time in which the circle ABC is described to the periodic time in which the circle DEF is described, HK will be to AG in the duplicate ratio of the periodic time in which the circle DEF is described to the periodic time in which the circle ABC is described; and therefore the centripetal force of the body in the circumference of the circle ABC will be to the centripetal force of the body in the circumference of the circle DEF in the duplicate ratio of the periodic time in the circle DEF to the periodic time in the circle ABC.

P R O P. VI. *Fig. 93.*

Let ABCD represent the orbit of the moon round the earth at T, S the sun; and supposing the orbit of the moon to be a circle, and to coincide with the plane of the ecliptic, it is required to determine the forces with which

which the sun disturbs the motion of the moon round the earth.

Join ST meeting the circle ABCD in A, C; and suppose the moon at any point L in its orbit; join SL, TL; in SL take LE, such, that the square of SL is to the square of ST as ST to LE; draw LF parallel and equal to ST; and complete the parallelogram EFLG.

Because the square of SL is to the square of ST as ST to LE, if ST represent the force of the earth to the sun, LE will represent the force of the moon to the sun; let the force LE be resolved into the forces LF, LG: because TS, FL, are equal and parallel, the forces TS, FL, do not disturb the motion of the moon and earth with respect to one another; there remains therefore only the force LG to be considered. Draw GN perpendicular to TL, meeting TL in N; and complete the parallelogram LNGO; the force LG may be resolved into the forces LO, LN, and the force LO may be altogether neglected, because, when the moon is at the same distance from the point

point A, its effect is quite opposite and equal, and exactly balances the other effect; so that this force does not affect the periodic time of the moon round the earth; there remains therefore only the force LN to be considered. If the point N falls without the circle, the action of the sun upon the moon diminishes the gravity of the moon to the earth; if the point N falls within the circle, the action of the sun upon the moon increases the gravity of the moon to the earth; and if the point N coincides with the point L, the action of the sun upon the moon has no effect.

In ST take TP on the same side the centre T with S; and let the rectangle STP be equal to the square of TA; bisect SP in Q, and ST in R; and in ST take TK, so, that ST may be triple of TK; let LM perpendicular to TA, meet TA in M; in TA take Tb, so, that the square of TA may be triple the square of Tb; and in TC take Te, equal to Tb; let LG meet ST in H; and draw HZ perpendicular to TL, meeting TL in Z.

Because the square of SL is to the square of

of ST as ST to LE; and the square of SL is [4.] equal to twice the rectangle contained by QM, ST, twice the rectangle contained by QM, ST, will be to the square of ST as ST to LE; that is, twice QM will be to ST as ST to LE; therefore QM will be to RT as ST to LE: but SH is to SL as ST, or EG, to EL; therefore QM will be to RT as SH to SL. Again, because the rectangle STP is equal to the square of TA, ST will be to TA as TA to TP: but because ST, the distance of the sun from the earth, is great in comparison of TA, the distance of the moon from the earth, TA will be great in comparison of TP; therefore TS, SP, may be considered as equal; and therefore QT, TR, may be considered as equal, and the points Q, R, as coinciding; therefore QM, MR, may be considered as equal; therefore MR will be to RT as HS to SL: but because TS is great in comparison of TA, or TL, the angle TSL will be little; therefore LS, SM, may be considered as equal; therefore MR will be to RT as HS to SM; therefore, inversely, TR will be to RM as MS to SH; therefore TM will be to MH as RT

to

Fig 90

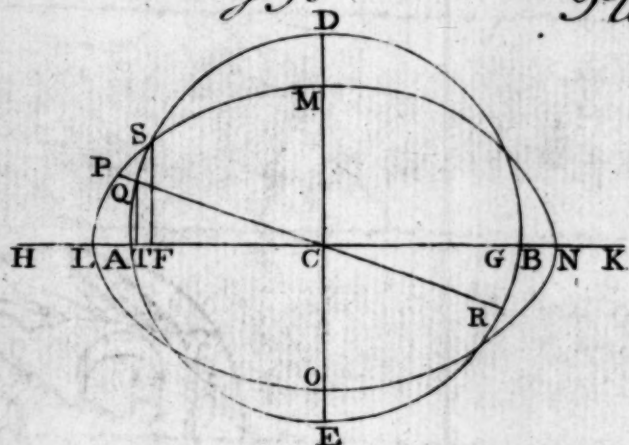


Plate. 14.

Fig. 91

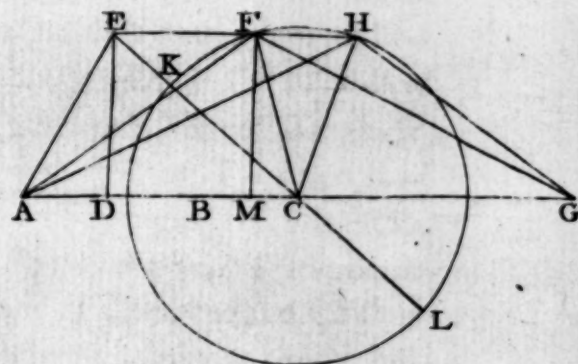


Fig 92

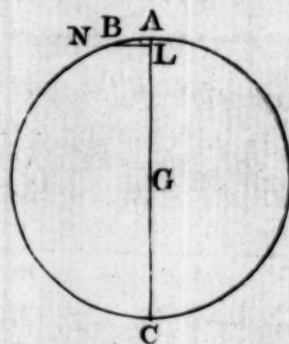
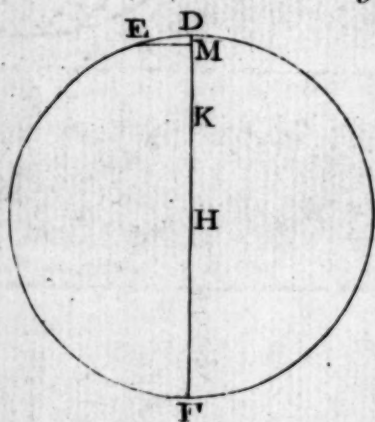
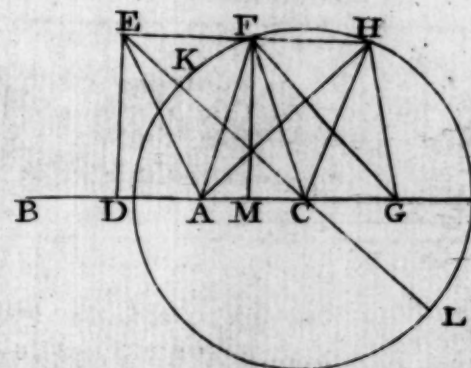


Fig. 91



to SM : but SM is nearly double of RT, therefore HM will be nearly double of TM; and therefore TH may be considered as triple of TM : because ST is triple of TK, and HT triple of TM, SH will be triple of KM ; therefore MK will be to KT as HS to ST, or EG ; that is, as HL to LG ; that is, as ZL to LN : but the rectangle TLZ is to the rectangle TLN as ZL to LN ; therefore MK is to KT as the rectangle TLZ to the rectangle TLN : and because the angles TZH, TML, are equal, because both right, the rectangle LTZ will be equal to the rectangle HTM ; that is, equal to thrice the square of TM : and because the square of TL, or TA, is triple the square of Tb, therefore the rectangle TLZ will be triple the rectangle eMb ; and therefore MK will be to KT as thrice the rectangle eMb to the rectangle TLN ; therefore MK will be to ST as thrice the rectangle eMb to thrice the rectangle TLN ; that is, as the rectangle eMb to the rectangle TLN : but the rectangle contained by MK, TL, is to the rectangle STL as MK to ST ; therefore the rectangle con-

Qq

tained

tained by MK , TL , will be to the rectangle STL as the rectangle eMb to the rectangle TLN ; and therefore, alternating, the rectangle contained by MK , TL , will be to the rectangle eMb as the rectangle STL to the rectangle TLN ; that is, as ST to LN : and because the centripetal force of the earth to the sun is to the disturbing force of the sun upon the moon at the point L as ST to LN ; therefore the centripetal force of the earth to the sun will be to the disturbing force of the sun upon the moon at the point L as the rectangle contained by MK , TL , to the rectangle eMb . *Q. E. I.*

CONSTRUCTION.

In ST take TK , so, that ST may be triple of TK ; and in TA take Tb , so, that the square of TA may be triple the square of Tb ; and in TC take Te equal to Tb ; suppose the moon at any point L in its orbit; draw LM perpendicular to TK , meeting TK in M ; the centripetal force of the earth to the sun will be to the disturbing force of the sun upon the moon at the point

point L , as the rectangle contained by MK , TL , to the rectangle eMb . If the point M falls without b , e , the disturbing force diminishes the gravity of the moon to the earth; but if the point M falls between b , e , the disturbing force increases the gravity of the moon to the earth; and if the point M coincides with either of the points b , e , the disturbing force has no effect.

COR. The force of the earth to the sun is to the solar force acting upon the moon in the direction LH nearly as ST to LH .

Because TH is little in comparison of ST , SH , ST , may be considered as equal; that is, SH , EG , may be considered as equal; therefore LH , LG , may be considered as equal: but ST represents the force of the earth to the sun, and LG represents the solar force acting upon the moon in the direction LH ; therefore the force of the earth to the sun is to the solar force acting upon the moon in the direction LH nearly as ST to LH . Q. E. D.

Q q 2

PROP.

PROP. VII. Fig. 93.

The same construction remaining as in the former proposition; and let TA be to TV in the duplicate ratio of the periodic time of the moon round the earth to the periodic time of the earth round the sun; the centripetal force of the moon to the earth will be to the disturbing force of the sun upon the moon at the point L nearly as the rectangle VTL or VTA to thrice the rectangle eMb.

Because LZ is to LN as LH to LG, that is, as SH to EG, or ST, and SH, ST, are nearly equal, because TH is little in comparison of ST; therefore LZ, LN, may be considered as equal; therefore the rectangle TLN will be nearly equal to thrice the rectangle eMb. Because TA is to TV in the duplicate ratio of the periodic time of the moon round the earth to the periodic time of the earth round the sun, the centripetal force of the moon to the earth will be to the centripetal force of the earth to the sun [5.] as TV to TS: but the centripetal

petal force of the earth to the sun is to the disturbing force of the sun upon the moon as ST to LN ; therefore the centripetal force of the moon to the earth will be to the disturbing force of the sun upon the moon as TV to LN : but TV is to LN as the rectangle VTL to the rectangle TLN ; that is, as the rectangle VTL to thrice the rectangle eMb ; therefore the centripetal force of the moon to the earth will be to the disturbing force of the sun upon the moon at the point L as the rectangle VTL to thrice the rectangle eMb . *Q. E. D.*

P R O P. VIII. *Fig. 94.*

Let ABCD represent the orbit of the moon round the earth at T, S the sun; and supposing the orbit of the moon to be a circle, and to coincide with the plane of the ecliptic; join ST meeting the orbit in A, C; in ST take TK, so, that ST may be triple of TK; produce TC to P, so, that TP, TK, may be equal; in TA take Tb, so, that the square of TA may be triple the square of Tb; and in TC take Te equal to Tb;

Tb; let L be any point in the orbit; draw TL meeting the orbit again in l ; draw LM perpendicular to AC , meeting AC in M ; and take LN , such, that the rectangle KMP may be to the rectangle eMb as the rectangle STK to the rectangle TLN : if ST represent the centripetal force of the earth to the sun, LN will represent half the sum of the disturbing forces of the sun upon the moon at the points L, l , in its orbit.

Draw lm perpendicular to AC , meeting AC in m ; and take MQ, mq , such, that the rectangle contained by KM, TA , may be to the rectangle eMb as ST to MQ ; and likewise the rectangle contained by Km, TA , may be to the rectangle eMb as ST to mq .

Because the rectangle contained by KM, TA , is to the rectangle eMb as ST to MQ , inversely, the rectangle eMb will be to the rectangle contained by KM, TA , as MQ to ST : but because the rectangle contained by Km, TA , is to the rectangle emb , or eMb , as ST to mq ; and the rectangle eMb is to the rectangle contained by KM, TA ,

as MQ to ST ; the rectangle contained by Km , TA , will be to the rectangle contained by KM , TA , as MQ to mq ; that is, Km will be to KM as MQ to mq ; therefore Km will be to the sum of Km , KM , as MQ to the sum of MQ , mq ; but the sum of Km , KM , is equal to twice KT , (because TM , Tm are equal); therefore Km is to twice KT as MQ to the sum of MQ , mq ; and therefore Km will be to MQ as twice KT to the sum of MQ , mq ; but the rectangle MKm is to the rectangle KMQ as Km to MQ ; therefore the rectangle MKm will be to the rectangle KMQ as twice KT to the sum of MQ , mq ; but the rectangle MKm is equal to the rectangle KMP , because Km , MP , are equal; therefore the rectangle KMP is to the rectangle KMQ as twice KT to the sum of MQ , mq ; that is, as twice the rectangle STK to the rectangle contained by ST , and the sum of MQ , mq .

Again, because the rectangle contained by KM , TA , is to the rectangle eMb as ST to MQ ; that is, as the rectangle contained by ST , KM , to the rectangle KMQ ;
alternating,

alternating, the rectangle contained by KM, TA, will be to the rectangle contained by ST, KM, as the rectangle eMb to the rectangle KMQ; therefore TA will be to ST as the rectangle eMb to the rectangle KMQ; and therefore the rectangle KMQ will be to the rectangle eMb as SF to TA, or TL; that is, as the rectangle contained by ST, and the sum of MQ, mq , to the rectangle contained by TL, and the sum of MQ, mq : but because the rectangle KMP is to the rectangle KMQ as twice the rectangle STK to the rectangle contained by ST, and the sum of MQ, mq ; and the rectangle KMQ is to the rectangle eMb as the rectangle contained by ST, and the sum of MQ, mq , to the rectangle contained by TL, and the sum of MQ, mq ; therefore, by equality, the rectangle KMP will be to the rectangle eMb as twice the rectangle STK to the rectangle contained by TL, and the sum of MQ, mq : but the rectangle KMP is to the rectangle eMb as the rectangle STK to the rectangle TLN, that is, as twice the rectangle STK to twice the rectangle TLN; therefore

twice

twice the rectangle STK will be to the rectangle contained by TL, and the sum of MQ, mq , as twice the rectangle STK to twice the rectangle TLN; therefore twice the rectangle TLN will be equal to the rectangle contained by TL, and the sum of MQ, mq ; and therefore twice LN is equal to the sum of MQ, mq : but because the rectangle contained by KM, TA, is to the rectangle eMb as ST to MQ, and likewise the rectangle contained by Km, TA, is to the rectangle emb as ST to mq ; therefore [6.] if ST represent the centripetal force of the earth to the sun, MQ, mq , will represent the disturbing forces of the sun upon the moon at the points L, l , in its orbit; therefore (because twice LN is equal to the sum of MQ, mq .) if ST represent the centripetal force of the earth to the sun, LN will represent half the sum of the disturbing forces of the sun upon the moon at the points L, l . Q. E. D.

R r

PROP.

PROP. IX. Fig. 95.

Let the circle $ABCD$ whose centre is T represent the orbit of the moon round the earth at T ; let AC be the syzigial line, and BD the line of the quadratures; in TA take Tb , so, that the square of TA may be triple the square of Tb ; and in TC take Te equal to Tb ; produce TA to K ; and let TK represent one third part of the distance of the sun from the earth; and produce TC to P ; and let TP be equal to TK ; in TK take TE , such, that the square of TE is to the square of TA as the rectangle KbP to the rectangle KAP ; and in TB take TF , such, that the square of TB is to the square of TF as the square of TK to the rectangle KbP ; with the semiaxes TE , TF , let there be an ellipse $EFGH$ described; take TQ such, that TK may be to TA as the square of TA to the square of TQ ; draw TL to any point L in the circle, meeting the ellipse in M , and the circle in l : the centripetal force of the earth to the sun will be to half the sum of the disturbing forces of the sun upon

upon the moon at the points L, l , as the square of TQ to the rectangle LMI .

Draw LN perpendicular to AC , meeting TA in N ; and let thrice the square of TK be to the rectangle TLO as the rectangle KNP to the rectangle bNe .

Because thrice the square of TK is to the rectangle TLO as the rectangle KNP to the rectangle bNe , and the square of TL or TA is [2.] to the rectangle LMI as the rectangle KNP to the rectangle bNe ; therefore thrice the square of TK will be to the rectangle TLO as the square of TL or TA to the rectangle LMI ; therefore, alternating, thrice the square of TK will be to the square of TL or TA as the rectangle TLO to the rectangle LMI . Again, because the square of TK is to the rectangle ATK as TK to TA ; that is, as the square of TA to the square of TQ ; alternating, the square of TK will be to the square of TA as the rectangle ATK to the square of TQ ; therefore thrice the square of TK will be to the square of TA as thrice the rectangle ATK to the square of TQ : but thrice

the square of TK is to the square of TA as the rectangle TLO to the rectangle $LM/$; therefore thrice the rectangle ATK will be to the square of TQ as the rectangle TLO to the rectangle $LM/$; and, alternating, thrice the rectangle ATK will be to the rectangle TLO as the square of TQ to the rectangle $LM/$; that is, (because TA , TL , are equal), thrice TK will be to LO as the square of TQ to the rectangle $LM/$: but the centripetal force of the earth to the sun is [8.] to half the sum of the disturbing forces of the sun upon the moon at the points L , l , as thrice TK to LO ; therefore the centripetal force of the earth to the sun is to half the sum of the disturbing forces of the sun upon the moon at the points L , l , as the square of TQ to the rectangle $LM/$.
Q. E. D.

P R O P. X. *Fig. 96.*

The same construction remaining as in the former proposition; let the ellipse EFGH meet the circle ABCD in the point S in the arc AB; the force of the earth to the sun

sun will be to the mean disturbing force of the sun upon the moon as the square of TQ to the excess of the rectangle ETF above the square of TA ; likewise the sum of the forces diminishing the gravity of the moon to the earth will be to the sum of the forces increasing the gravity of the moon to the earth as the space ASE to the space BSF .

Let the arc AS be divided into indefinitely little equal parts $Ad, df, fg, gh, \&c.$; and join $Td, Tf, Tg, Th, \&c.$; meeting the arc ES of the ellipse in $k, l, m, n, \&c.$ and the circle in $o, p, q, \&c.$; with the centre T , and distance TQ , describe the circle QLM meeting $TA, Td, Tf, Tg, Th, \&c.$ in $Q, r, s, t, v, \&c.$; and let the ellipse meet the arcs BC, CD, DA , in O, R, V .

Because the sector ETk is to the sector ATd as the square of TE to the square of TA ; by division, the space $EAdk$ will be to the sector ATd as the rectangle AEC to the square of TA : but the sector ATd is to the sector QTr as the square of TA to the square of TQ ; therefore, by equality, the space $EAdk$ will be to the sector QTr

as

as the rectangle AEC to the square of TQ ; therefore, inversely, the sector QTr will be to the space $EAdk$ as the square of TQ to the rectangle AEC : but the force of the earth to the sun is to half the sum of the disturbing forces of the sun upon the moon at the points A, C , [9.] as the square of TQ to the rectangle AEC ; therefore the force of the earth to the sun will be to half the sum of the disturbing forces of the sun upon the moon at the points A, C , as the sector QTr to the space $EAdk$; therefore, if the sector QTr represent the force of the earth to the sun, the space $EAdk$ will represent half the sum of the disturbing forces of the sun upon the moon at the points A, C . The same way it is shown, that if the sector rTs represent the force of the earth to the sun, the space $kdfi$ will represent half the sum of the disturbing forces of the sun upon the moon at the points d, o ; likewise the space $lfgm$ will represent half the sum of the disturbing forces of the sun upon the moon at the points f, p ; and the space $mgbn$ will represent half the sum of the disturbing forces of the sun upon the moon at the points g, q , and so on; therefore

therefore it is evident, that the space ASE will represent half the sum of the disturbing forces of the sun acting upon the moon in its progress from A to S, and from C to R; therefore the space contained by the arc SEV of the ellipse, and the arc SAV of the circle, will represent the sum of the disturbing forces of the sun acting upon the moon in its progress from A to S, and from C to R. The same way it is shown, that the space contained by the arc OGR of the ellipse, and the arc OCR of the circle, will represent the sum of the disturbing forces of the sun acting upon the moon in its progress from V to A, and from O to C; therefore the portion of the ellipse that falls without the circle ABCD will represent the sum of the disturbing forces of the sun acting upon the moon in its progress through the arcs VAS, OCR. The same way it is shown, that the portion of the circle that falls without the ellipse will represent the sum of the disturbing forces of the sun acting upon the moon in its progress through the arcs SBO, VDR: but the action of the sun upon the moon diminishes the gravity of the

the moon to the earth in its progress through the arcs VAS, OCR, and increases the gravity of the moon to the earth in its progress through the arcs SBO, VDR; therefore the portion of the ellipse that falls without the circle ABC will represent the sum of the disturbing forces diminishing the gravity of the moon to the earth; and the portion of the circle that falls without the ellipse will represent the sum of the disturbing forces increasing the gravity of the moon to the earth.

Again, because the rectangle ABC is not less than the square of Tb, the rectangle ETF will [3.] be greater than the square of TA: but the ellipse EFGH is to the circle ABCD as the rectangle ETF to the square of TA; therefore the ellipse EFGH is greater than the circle ABCD; and therefore the portion of the ellipse that falls without the circle ABCD will be greater than the portion of the circle that falls without the ellipse; therefore the sum of the forces diminishing the gravity of the moon to the earth is greater than the sum of the forces increasing the gravity of the moon to the earth;

earth; and therefore, upon the whole, the action of the sun upon the moon diminishes the gravity of the moon to the earth; and therefore the excess of the portion of the ellipse that falls without the circle ABCD above the portion of the circle that falls without the ellipse, will represent the sum of the forces by which the gravity of the moon to the earth is diminished during the periodic revolution of the moon round the earth; that is, the excess of the ellipse EFGH above the circle ABCD will represent the sum of the forces by which the gravity of the moon to the earth is diminished during the periodic revolution of the moon round the earth: but it is evident from what was shown above, that the circle QLM will represent the sum of the solar forces that have acted upon the earth during the periodic revolution of the moon round the earth; therefore the force of the earth to the sun, will be to the mean disturbing force of the sun upon the moon, as the circle QLM to the excess of the ellipse EFGH above the circle ABCD: but because the ellipse EFGH is to the circle

S f ABCD

ABCD as the rectangle ETF to the square of TA; by division, the excess of the ellipse EFGH above the circle ABCD, will be to the circle ABCD, as the excess of the rectangle ETF above the square of TA is to the square of TA: but the circle ABCD is to the circle QLM as the square of TA is to the square of TQ; therefore, by equality, the excess of the ellipse EFGH above the circle ABCD, will be to the circle QLM, as the excess of the rectangle ETF above the square of TA is to the square of TQ; therefore, inversely, the circle QLM will be to the excess of the ellipse above the circle ABCD, as the square of TQ to the excess of the rectangle ETF above the square of TA: but the force of the earth to the sun is to the mean disturbing force of the sun upon the moon, as the circle QLM to the excess of the ellipse above the circle ABCD; therefore the centripetal force of the earth to the sun, will be to the mean disturbing force of the sun upon the moon, as the square of TQ to the excess of the rectangle ETF above the square of TA.

Again, because the portion of the ellipse
that

that falls without the circle ABCD represents the sum of the disturbing forces diminishing the gravity of the moon to the earth, and the portion of the circle ABCD that falls without the ellipse represents the sum of the disturbing forces increasing the gravity of the moon to the earth; therefore the sum of the forces diminishing the gravity of the moon to the earth, will be to the sum of the forces increasing the gravity of the moon to the earth, as the portion of the ellipse that falls without the circle ABCD to the portion of the circle ABCD that falls without the ellipse: but because the portion of the ellipse that falls without the circle ABCD is quadruple of the space ASE, and the portion of the circle ABCD that falls without the ellipse is quadruple of the space BSF; therefore the portion of the ellipse that falls without the circle ABCD, will be to the portion of the circle ABCD that falls without the ellipse, as the space ASE to the space BSF; therefore the sum of the forces diminishing the gravity of the moon to the earth, will be to the sum of the forces increasing the gravity of the moon to the earth,

earth, as the space ASE, to the space BSF.

Q. E. D.

PROP. XI. Fig. 97.

If from any point A without a circle there be drawn a line to C the centre of the circle, and likewise there be drawn AD a tangent to the circle at D, the excess of the square of AC above half the square of the semidiameter of the circle will be greater than the rectangle CAD.

Join CD, and draw DE perpendicular to AC, meeting AC in E; bisect EC in F; and in AC take AG equal to AD.

Because AD is a tangent to the circle at D, the angle ADC is a right angle; therefore AC is to AD as AD to AE; therefore AC is to AG as AG to AE; therefore AC is to AG as CG to GE; but AC is greater than AD or AG; therefore CG is greater than GE; therefore CE is greater than twice GE; therefore FE the half of CE is greater than EG; therefore FA is greater than AG or AD; therefore the rectangle

CAF

CAF is greater than the rectangle CAD : but because AC is to CD as CD to CE, the rectangle ACE is equal to the square of CD; therefore the rectangle ACF is equal to half the square of CD : but the rectangle CAF is the excess of the square of AC above the rectangle ACF; that is, the rectangle CAF is the excess of the square of AC above half the square of CD; and the rectangle CAF is greater than the rectangle CAD; therefore the excess of the square of AC above half the square of the semidiameter CD is greater than the rectangle CAD. Q. E. D.

P R O P. XII. Fig. 98.

From any point A without a circle let there be drawn AB a tangent to the circle at B, and likewise let there be drawn AC to the centre C; in AC take AG, such, that the rectangle GAB may be equal to the excess of the square of AC, above one third part of the square of the semidiameter of the circle; six time the rectangle contained by AB, CG, will be greater than the square of the semidiameter of the circle.

From

From B let fall BD perpendicular to AC, meeting AC in D; in CD take the point E, such, that CD may be triple of CE; in AC take AF equal to AB; join CB. Because the rectangle GAB is equal to the excess of the square of AC above one third part of the square of the semidiameter of the circle, and the excess of the square of AC above one third part of the square of the semidiameter of the circle is greater than the excess of the square of AC above half the square of the semidiameter of the circle; therefore the rectangle GAB is greater than the excess of the square of AC above half the square of the semidiameter of the circle: but the excess of the square of AC above half the square of the semidiameter of the circle is [P. I.] greater than the rectangle CAB; therefore the rectangle GAB is greater than the rectangle CAB; therefore GA is greater than AC. Again, because AC is to CB as CB to CD, the rectangle ACD will be equal to the square of CB; therefore (because CD is triple of CE) the rectangle ACE is one third part of the square of CB; therefore the rectangle GAB is equal

qual to the rectangle CAE; therefore AB or AF is to AE as AC to AG; therefore AF is to FE as AC to CG; therefore the rectangle contained by AF, CG, is equal to the rectangle contained by AC, FE; that is, the rectangle contained by AB, CG, is equal to the rectangle contained by AC, FE. Again, because AC is to AF as AF to AD, AC will be to AF as CF to FD; and because AC is greater than AF; therefore CF will be greater than FD; therefore twice CF will be greater than CD; that is, greater than thrice CE; and therefore twice FE will be greater than EC; therefore six times FE will be greater than CD; therefore six times the rectangle contained by AC, FE, will be greater than the rectangle ACD; that is, greater than the square of CB; and therefore six times the rectangle contained by AB, CG, is greater than the square of the semidiameter.

COR. Six times the rectangle ACG is greater than the square of the semidiameter.

P. R. O. P.

PROP. XIII. Fig. 99.

Let the circle $ABCD$ whose centre is T , represent the orbit of the moon round the earth at T ; let AC be the syzigial line; in TA produced take TK equal to one third part of the distance of the sun from the earth; draw KE a tangent to the circle at E ; in KT produced toward T take KF , such, that the rectangle FKE may be equal to the excess of the square of KT above one third part of the square of TA : the force of the earth to the sun will be to the mean disturbing force of the sun upon the moon as AT to TF .

In TA take Tb , so, that the square of TA may be triple the square of Tb ; and produce TC to P ; and let TP be equal to TK ; in TA take TG , such, that the square of TG may be to the square of TA as the rectangle KbP to the rectangle KAP ; and in TB take TH , such, that the square of TB may be to the square of TH as the square of TK to the rectangle KbP ; let the

Fig. 93

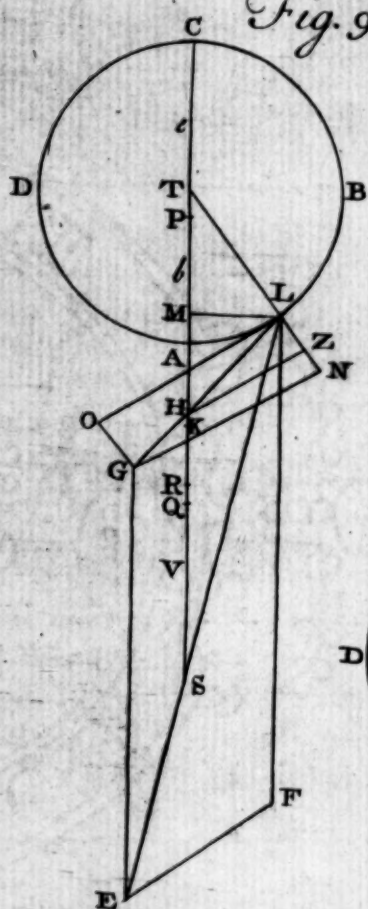


Fig. 95 P.328.

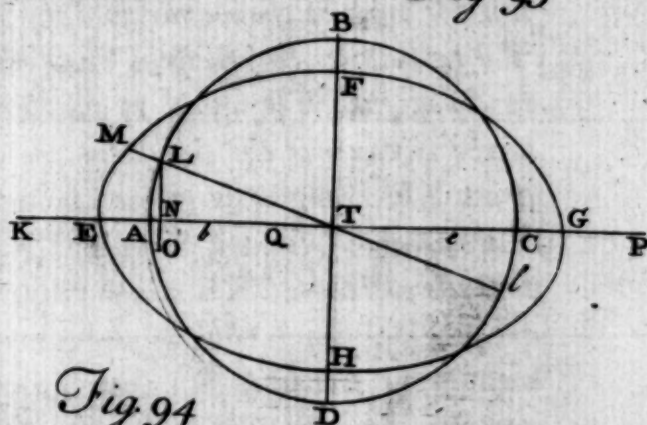


Fig. 94

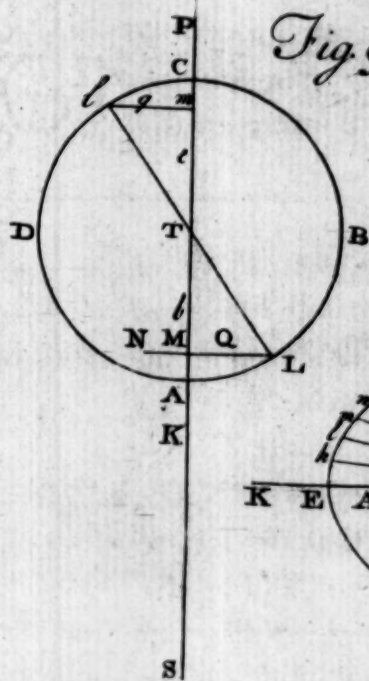
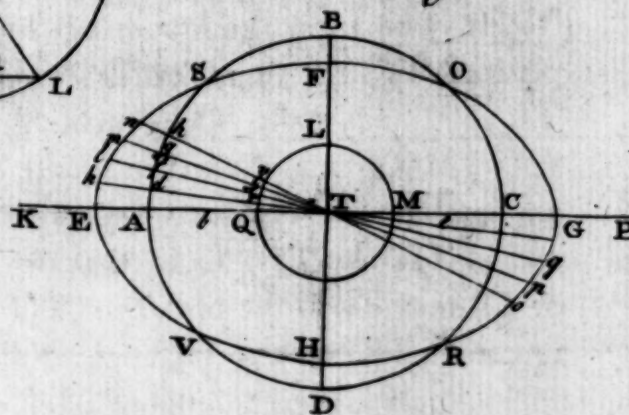


Fig. 96



the square of TA be to the square of TQ as TK to TA.

Because the square of TA is triple the square of Tb, the rectangle K δ P will be equal to the excess of the square of TK above one third part of the square of TA; that is, equal to the rectangle FKE: and because KC, AP, are equal, the rectangle KAP will be equal to the rectangle AKC; that is, equal to the square of KE: but the square of TG is to the square of TA as the rectangle K δ P to the rectangle KAP; therefore the square of TG will be to the square of TA as the rectangle FKE to the square of KE; that is, as FK to KE; that is, as the square of FK to the rectangle FKE; therefore TG will be to TA in the subduplicate ratio of the square of FK to the rectangle FKE: but the rectangle GTH is to the rectangle ATH as GT to TA; therefore the rectangle GTH will be to the rectangle ATH in the subduplicate ratio of the square of FK to the rectangle FKE. Again, because the square of TH is to the square of TB, or TA, as the rectangle K δ P, or FKE, to the square of KT;

T t therefore

therefore TH will be to TA in the subduplicate ratio of the rectangle FKE to the square of KT : but the rectangle ATH is to the square of TA as TH to TA ; therefore the rectangle ATH is to the square of TA in the subduplicate ratio of the rectangle FKE to the square of KT : but because the rectangle GTH is to the rectangle ATH in the subduplicate ratio of the square of FK to the rectangle FKE, and the rectangle ATH is to the square of TA in the subduplicate ratio of the rectangle FKE to the square of KT ; therefore, by equality, the rectangle GTH will be to the square of TA in the subduplicate ratio of the square of FK to the square of KT ; therefore the rectangle GTH is to the square of TA as FK to KT ; therefore, by division, the excess of the rectangle GTH above the square of TA will be to the square of TA as FT to TK : but the square of TA is to the square of TQ as TK to TA ; therefore, by equality, the excess of the rectangle GTH above the square of TA will be to the square of TQ as TF to TA ; therefore, inversely, the square of TQ will be to the
excess

excess of the rectangle GTH above the square of TA as TA to TF: but the force of the earth to the sun is [10.] to the mean disturbing force of the sun upon the moon as the square of TQ to the excess of the rectangle GTH above the square of TA; therefore the force of the earth to the sun will be to the mean disturbing force of the sun upon the moon as TA to TF. Q. E. D.

PRO P. XIV. Fig. 100.

Let the circle ABCD represent the orbit of the moon round the earth at the centre T; in TA take TK equal to one third part of the distance of the sun from the earth, and draw KE a tangent to the circle at E; produce KT to F, so, that the rectangle FKE may be equal to the excess of the square of KT above one third part of the square of TA; let TA be to TV in the duplicate ratio of the periodic time of the moon round the earth to the periodic time of the earth round the sun: the force of the moon to the earth will be to the mean disturbing force of

the sun upon the moon as the rectangle ATV to thrice the rectangle KTF.

Because TA is to TV in the duplicate ratio of the periodic time of the moon round the earth to the periodic time of the earth round the sun, the force of the earth to the sun will be [5.] to the force of the moon to the earth as thrice TK to TV; that is, as thrice the rectangle ATK to the rectangle ATV; therefore, inversely, the force of the moon to the earth will be to the force of the earth to the sun as the rectangle ATV to thrice the rectangle ATK; and the force of the earth to the sun is to the mean disturbing force of the sun upon the moon as TA to TF [13.]; that is, as thrice the rectangle ATK to thrice the rectangle KTF; therefore, by equality, the force of the moon to the earth will be to the mean disturbing force of the sun upon the moon as the rectangle ATV to thrice the rectangle KTF.

Q. E. D.

P R O P.

P R O P. XV. Fig. 100.

The force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon in a less ratio than the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth.

The same construction remaining as in the former proposition.

Because the force of the moon to the earth is to the mean disturbing force of the sun upon the moon as the rectangle ATV to thrice the rectangle KTF , the force of the moon to the earth will be to the double of the mean disturbing force of the sun upon the moon as the rectangle ATV to six times the rectangle KTF : but because six times the rectangle KTF is [cor. to prop. 12.] greater than the square of TA , the ratio of the rectangle ATV to six times the rectangle KTF will be less than the ratio of the rectangle ATV to the square of TA ; that is,

less

less than the ratio of TV to TA: but the force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon as the rectangle ATV to six times the rectangle KTF, and TV is to TA in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; therefore the force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon, in a less ratio than the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth.

Q. E. D.

PROP. XVI. Fig. 101.

Let there be a circle whose diameter is AB, and centre C; let DE, a tangent to the circle at E, meet AB in D; draw CN parallel to DE, meeting the circle in N, and draw NK perpendicular to AB, meeting AB in K; in DB take DF, so, that the rectangle FDE may be equal to the excess of the square of CD above one third part of the square of AC; in CA take

take CL, so, that the rectangle ACL may be triple the rectangle DCF, and in CD take CM, so, that the rectangle MCL may be equal to the square of AC: the rectangle MKA will be triple of the rectangle MCA.

Draw EG perpendicular to AB, meeting AB in G; in CG take CH, so, that CG may be triple of CH; and join CE.

Because CG is triple of CH, the rectangle DCG will be triple the rectangle DCH: but the rectangle DCG is equal to the square of CA; therefore the square of CA is triple the rectangle DCH; therefore the rectangle CDH will be equal to the excess of the square of CD above one third part of the square of CA; therefore the rectangle FDE is equal to the rectangle CDH; therefore FD is to DH as DC to DE; that is, as CN or CB to CK; therefore, conversely, DF is to FH as CB to BK. Again, because the rectangle MCL, that is, the square of AC, is triple of the rectangle DCH, and the rectangle ACL is triple of the rectangle DCF, the rectangle DCH will be to the rectangle DCF as the rectangle MCL to the rectangle ACL;

ACL; therefore CH is to CF as MC to CA or CB; therefore, by composition, HF is to FC as MB to BC. Because DF is to FH as CB to BK, and FH to FC as MB to BC; therefore DF is to FC as MB to BK; therefore DC is to CF as MK to KB: but DC is to CF as the square of DC to the rectangle DCF, and MK is to KB as the rectangle MKA to the rectangle AKB; therefore the square of DC is to the rectangle DCF as the rectangle MKA to the rectangle AKB; that is, as the rectangle MKA to the square of KN; therefore the rectangle MKA is to the square of KN as the square of DC to the rectangle DCF: but the square of KN is to the square of CN or CA as the square of CE or CA to the square of CD; therefore the rectangle MKA is to the square of KN as the square of DC to the rectangle DCF: but the square of KN is to the square of CN or CA as the square of CE or CA to the square of CD; therefore the rectangle MKA is to the square of CA as the square of CA to the rectangle DCF; therefore the rectangle MKA is to thrice the square of CA as the square of CA to thrice the rectangle

angle

angle DCF: but because the rectangle MCL is equal to the square of CA, and the rectangle ACL triple of the rectangle DCF; therefore the square of CA is to thrice the rectangle DCF as the rectangle MCL to the rectangle ACL; that is, as MC to CA; that is, as the rectangle MCA to the square of CA; therefore the rectangle MKA is to thrice the square of CA as the rectangle MCA to the square of CA; and therefore the rectangle MKA is triple of the rectangle MCA. Q. E. D.

P R O P. XVII. Fig. 102.

Let there be a circle whose diameter is AB, and centre C; and let DE a tangent to the circle at E meet AB in D; in DC take DF, such, that the rectangle FDE may be equal to the excess of the square of DC above one third part of the square of AC: if CD be great in comparison of AC, six times the rectangle DCF will be nearly equal to the square of AC.

Draw EG perpendicular to AC, meeting
U u AC

AC in G, and join CE; in CG take CH, such, that CG is triple of CH; and in CB take CK, equal to EG; draw KN perpendicular to CB, meeting the circle in N; and join CN.

Because DE is a tangent to the circle at E, and EG is perpendicular to AC, the rectangle DCG will be equal to the square of AC; and because CG is triple of CH, the rectangle DCH will be one third part of the square of AC; therefore the rectangle CDH will be the excess of the square of CD above one third part of the square of AC; therefore the rectangle FDE will be equal to the rectangle CDH; therefore FD will be to DH as DC to DE: and because DC is great in comparison of CA or CE, CD and DE will be nearly equal; therefore FD and DH will be nearly equal; and therefore FD and DC will be nearly equal; therefore the rectangle contained by DF, CG, will be nearly equal to the rectangle DCG; that is, equal to the square of AC. Again, because FD is to DH as DC to DE; that is, as CE to EG; that is, as CB to CK; DF will be to FH as CB

to

to BK ; that is, the rectangle contained by DF, CG, will be to the rectangle contained by FH, CG, as the square of CB to the rectangle CBK : but the rectangle contained by DF, CG, is nearly equal to the square of CB or CA ; therefore the rectangle contained by FH, CG, will be nearly equal to the rectangle CBK ; therefore twice the rectangle contained by FH, CG, will be nearly equal to the rectangle ABK : but because CB, CK, are nearly equal, the rectangle ABK will be nearly equal to the rectangle AKB ; that is, nearly equal to the square of KN ; therefore twice the rectangle contained by FH, CG, will be nearly equal to the square of KN : but because CE, EG, are equal to CN, CK, and the angles EGC, CKN, equal, because both right ; therefore CG, KN, are equal ; therefore the squares of CG, KN, will be equal ; therefore twice the rectangle contained by FH, CG, will be nearly equal to the square of CG ; therefore twice FH will be nearly equal to CG ; therefore twice CF will be nearly equal to CH ; therefore six times the rectangle DCF will be nearly equal to the

U u 2

rectangle

rectangle DCG; that is, nearly equal to the square of AC. *Q. E. D.*

P R O P, XVIII. *Fig. 103.*

The mean disturbing forces of the sun upon the moon at different distances of the moon from the earth, are to one another nearly as the distances of the moon from the earth.

Let TK represent one third part of the distance of the sun from the earth at T, and let TL, Tl, represent two different distances of the moon from the earth; with the centre T, and distances TL, Tl, describe the circles ABL, *abl*; draw KB, *Kb*, tangents to the circles ABL, *abl*, at B, *b*; in KT produced towards T take KD, *Kd*, such, that the rectangle DKB is equal to the excess of the square of KT above one third part of the square of TA; and likewise the rectangle *dKb* equal to the excess of the square of KT above one third part of the square of Ta.

Because KT is great in comparison of TA, six times the rectangle KTD will [17.]

be

be nearly equal to the square of TA ; therefore TA will be to TD nearly as six times KT to TA ; but if the moon had described the circle ABL round the earth at T , the force of the earth to the sun would have been to the mean disturbing force of the sun upon the moon [13.] as TA to TD ; therefore the force of the earth to the sun would have been to the mean disturbing force of the sun upon the moon describing the circle ABL nearly as six times KT to TA . The same way it is shown, that if the moon had described the circle abl round the earth at T , the force of the earth to the sun would have been to the mean disturbing force of the sun upon the moon describing the circle abl nearly as six times KT to Ta . Because the force of the earth to the sun is to the mean disturbing force of the sun upon the moon describing the circle ABL nearly as six times KT to TA ; inversely, the mean disturbing force of the sun upon the moon describing the circle ABL , will be to the force of the earth to the sun nearly as TA to six times KT : but the force of the earth to the sun is to the mean disturbing force

force of the sun upon the moon describing the circle *abl* nearly as six times KT to Ta ; therefore, by equality, the mean disturbing force of the sun upon the moon describing the circle ABL will be to the mean disturbing force of the sun upon the moon describing the circle *abl* nearly as TA to Ta ; that is, nearly as TL to Tl ; therefore the mean disturbing force of the sun upon the moon at the distance TL from the earth, will be to the mean disturbing force of the sun upon the moon at the distance Tl from the earth, nearly as TL to Tl . $\mathcal{Q}$. *E. D.*

P R O P. XIX. *Fig. 104.*

The force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon, nearly in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth.

Let the circle $ABCD$ represent the orbit of the moon round the earth at the centre T ; in TA produced take TK equal to one third

third part of the distance of the sun from the earth, and draw KE a tangent to the circle at E; produce KT to F, so, that the rectangle FKE may be equal to the excess of the square of KT above one third part of the square of TA; let TA be to TV in the duplicate ratio of the periodic time of the moon round the earth to the periodic time of the earth round the sun.

Because the force of the moon to the earth is to the mean disturbing force of the sun upon the moon [14.] as the rectangle ATV to thrice the rectangle KTF, the force of the moon to the earth will be to the double of the mean disturbing force of the sun upon the moon as the rectangle ATV to six times the rectangle KTF: but because TK is great in comparison of TA, six times the rectangle KTF will [17.] be nearly equal to the square of TA; therefore the rectangle ATV will be to six times the rectangle KTF nearly as the rectangle ATV to the square of TA; that is, nearly as TV to TA; therefore the force of the moon to the earth will be to the double of the mean disturbing force of the sun upon the moon nearly

nearly as TV to TA : but TV is to TA in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth ; therefore the force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon, nearly in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth. *Q. E. D.*

P R O P. XX. *Fig. 105.*

Let AB be bisected in C, and in CA produced take AD equal to AC ; let E be a point in AD, and let DE be little in comparison of AD ; in AB, take the point F, such, that the rectangle AFE is equal to the rectangle contained by CE, BD : DE will be to FB nearly as five to one.

Because CE is less than CD, the ratio of BC to CE will be greater than the ratio of BC to CD ; therefore, by composition, the ratio of BE to EC will be greater than the ratio of BD to DC ; therefore the rectangle
contained

Plate. 16.

Fig 97

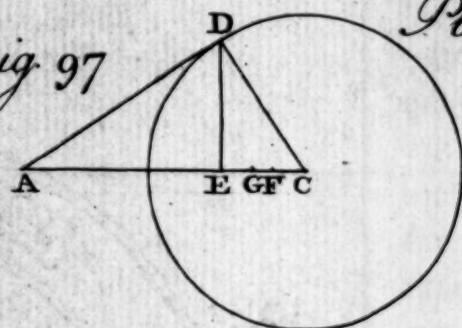


Fig. 98

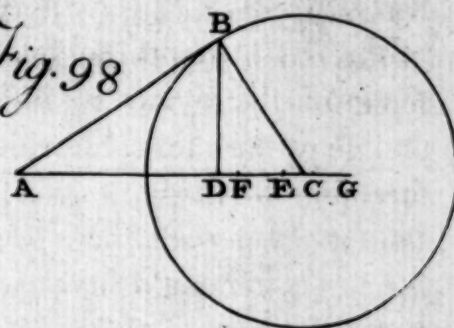


Fig 99

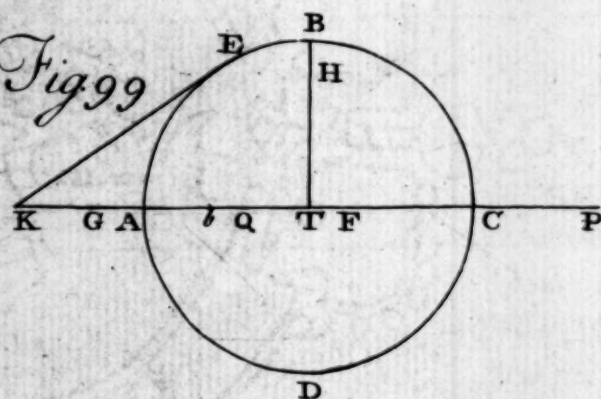


Fig. 100

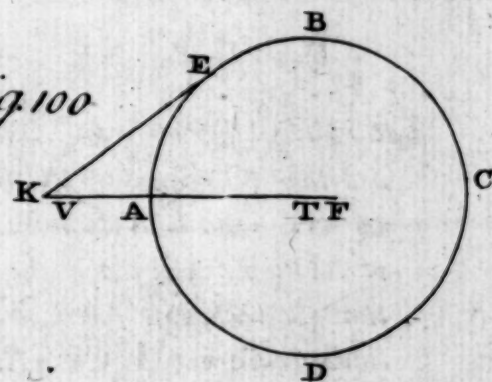


Fig 101

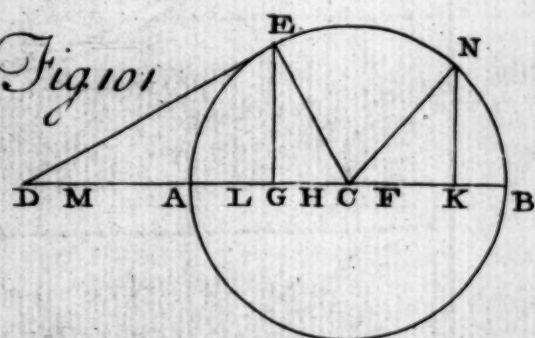
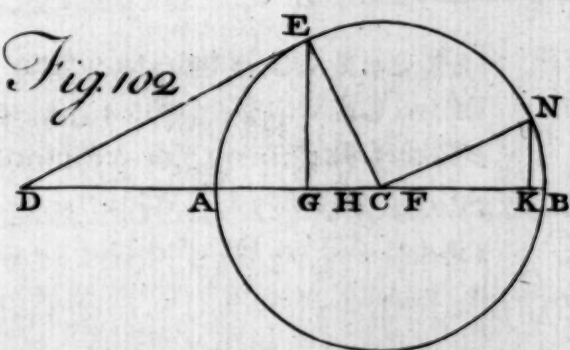


Fig. 102



contained by BE, DC, will be greater than
 the rectangle contained by CE, BD; that
 is, the rectangle ABE will be greater than
 the rectangle contained by CE, BD: but
 the rectangle AFE is equal to the rectangle
 contained by CE, BD; therefore the rect-
 angle ABE is greater than the rectangle
 AFE; therefore AB is greater than AF;
 and therefore BD is greater than EF. A-
 gain, because the rectangle AFE is equal to
 the rectangle contained by BD, CE, BD
 will be to EF as AF to CE; and because
 BD is greater than EF, therefore AF is
 greater than CE. Again, because BD is to
 EF as AF to CE, BD will be to its excess
 over EF as AF to its excess over CE;
 that is, BD will be to the sum of DE, BF,
 as AF to the excess of CF above AE: but
 the excess of CF above AE is equal to the
 excess of CB or AD above the sum of AE,
 BF; that is, equal to the excess of DE a-
 bove BF; therefore BD will be to the sum
 of DE, BF, as AF to the excess of DE a-
 bove BF; and, alternating, BD will be to
 AF as the sum of DE, FB, to the excess of
 DE above FB: but because DE is little in
 X x comparison

comparison of AD, and DE is greater than FB; therefore FB will be little in comparison of AD; that is, BF will be little in comparison of CB; therefore AF, AB, will be nearly equal; therefore DB will be to BA nearly as the sum of DE, BF, to the excess of DE above BF; therefore the sum of DB, BA, will be to the excess of DB above BA nearly as twice DE to twice BF; that is, as DE to BF; therefore the sum of DB, BA, will be to DA nearly as DE to FB; that is, DE will be to FB nearly as the sum of DB, BA, to DA: but because BD is triple of DA, the sum of DB, BA, will be to DA as five to one; therefore DE will be to FB nearly as five to one.

2. E, D.

COR. 1. In EB take Eg , so, that the rectangle contained by AB, Eg , may be equal to the rectangle contained by BD, CE: EF together with AB will be to AB as Bg to BF.

Because the rectangle AFE is equal to the rectangle contained by BD, CE; that is, equal to the rectangle contained by AB, Eg ;

Eg ; therefore EF will be to Eg as AB to AF ; therefore, conversely, EF will be to Fg as AB to BF ; therefore, alternating, EF will be to AB as qF to FB ; therefore, by composition, EF together with AB will be to AB as Bq to BF .

COR. 2. The ratio of Bq to BF is greater than the ratio of Eg together with AB to AB , but less than the ratio of EB together with AB to AB .

P R O P. XXI. Fig. 106.

Supposing the ratio of the force of the moon to the earth to the mean disturbing force of the sun upon the moon to be given, the ratio of the mean distance of the sun from the earth to the mean distance of the moon from the earth will be given.

Let the circle whose diameter is AB and centre T represent the orbit of the moon round the earth at T , and in TA produced take TK equal to one third part of the distance of the sun from the earth; draw

$X \times 2$

KE

KE a tangent to the circle at E; In KB take KP, such, that the rectangle FKE may be equal to the excess of the square of KT above one third part of the square of TA, and let TV be to TA in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; In AT produced take CT, so, that TV may be to TC as the force of the moon to the earth to the mean disturbing force of the sun upon the moon.

Because the rectangle VTA is [14.] to thrice the rectangle KTF as the force of the moon to the earth to the mean disturbing force of the sun upon the moon; that is, as VT to TC; that is, as the rectangle VTA to the rectangle ATC; therefore thrice the rectangle KTF will be equal to the rectangle ATC: and because TV is to TA in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth, the ratio of TV to TA will be given: and because TA is given, therefore TV will be given. Again, because TV is to TC as the force of the moon to the earth to the mean disturbing
ing

ing force of the sun upon the moon, and by the supposition the force of the moon to the earth is to the mean disturbing force of the sun upon the moon in a given ratio; therefore TV is to TC in a given ratio: and because TV is given, therefore TC will be given; and therefore, because TA is given, the rectangle ATC will be given; therefore thrice the rectangle KTF will be given; therefore the rectangle KTF will be given; and therefore [16.] TK will be given; therefore the ratio of TA to TK will be given; and therefore the ratio of TA to thrice TK will be given; therefore the ratio of the mean distance of the moon from the earth to the mean distance of the sun from the earth will be given.

P R O P. XXII. Fig. 106.

The same construction remaining as in the former proposition, let the rectangle CTD be equal to the square of TA; in AB take the point G, such, that the rectangle DGA may be triple the rectangle DTA: the mean distance of the moon from the earth will be
to

to one third part of the distance of the sun from the earth in the subduplicate ratio of the rectangle AGB to the square of TA.

Draw GH perpendicular to AB, meeting the circle in H; and join TH, TE; and draw EL perpendicular to AB, meeting AB in L.

It is evident, from what was shown in prop. 16. that EL, TG, are equal: and because TE, TH, are equal, and the angles TLE, TGH, are equal; therefore the triangles TLE, TGH, will be every way equal, and equiangular: but the triangles KTE, TLE, are equiangular; therefore the triangles KTE, TGH, are equiangular; therefore TE or TA is to TK as GH to HT or TA; therefore the square of TA will be to the square of TK as the square of GH to the square of TA: but the square of GH is equal to the rectangle AGB, therefore the square of TA is to the square of TK as the rectangle AGB to the square of TA; therefore TA is to TK in the subduplicate ratio of the rectangle AGB to the square of TA: but TA is to TK as the
mean

mean distance of the moon from the earth to one third part of the distance of the sun from the earth ; therefore the mean distance of the moon from the earth is to one third part of the distance of the sun from the earth in the subduplicate ratio of the rectangle AGB to the square of TA.

COR. Let AB be any given line ; bisect AB in T ; let TV be to TA in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth ; and in TA take TC, such, that TV is to TC as the force of the moon to the earth to the mean disturbing force of the sun upon the moon ; in TA produced take TD, such, that the rectangle DTC may be equal to the square of TA ; in AB take the point G, such, that the rectangle DGA may be triple the rectangle DTA : the mean distance of the moon from the earth will be to one third part of the distance of the sun from the earth in the subduplicate ratio of the rectangle AGB to the square of TA.

PROP,

P R O P. XXIII. Fig. 107.

Let DC be any given line bisected in A ; and in AC produced towards C take CB equal to AC , and bisect BC in E ; take the line p , such, that AC may be to the line p in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; and in CD take CF , such, that CF may be to the line p as the force of the moon to the earth to the mean disturbing force of the sun upon the moon: the mean distance of the moon from the earth will be to one third part of the distance of the sun from the earth nearly in the subduplicate ratio of DF to DE .

Let CV be to CA as CA to p , and VC to CG as CF to p ; and in AB take the point H , such, that the rectangle FHA may be triple the rectangle FCA .

Because CF is to p as the force of the moon to the earth to the mean disturbing force of the sun upon the moon, therefore CF will be to the double of p as the force of the

the moon to the earth to the double of the mean disturbing force of the sun upon the moon: but [15.] the force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon in a less ratio than the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; therefore the ratio of CF to the double of p will be less than the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth: but CA is to p in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; therefore the ratio of CF to the double of p will be less than the ratio of CA to p ; that is, less than the ratio of CD to the double of p ; therefore CD is greater than CF. Again, because [19.] the force of the moon to the earth is to the double of the mean disturbing force of the sun upon the moon, nearly in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; therefore FC will be to p nearly as DC to p ; therefore

Y y

DC,

DC, CF, will be nearly equal; and therefore DF will be little in comparison of AD. Again, because VC is to CA as CA to p , the square of CA will be equal to the rectangle contained by CV, p ; and because VC is to CG as CF to p , the rectangle contained by VC, p , will be equal to the rectangle FCG; therefore the rectangle FCG is equal to the square of CA: and because the rectangle FHA is triple of the rectangle FCA, therefore the mean distance of the moon from the earth will [22.] be to one third part of the distance of the sun from the earth in the subduplicate ratio of the rectangle AHB to the square of CA. Again, because DB is triple of AC, the rectangle contained by DB, FC, will be triple the rectangle FCA; therefore the rectangle FHA will be equal to the rectangle contained by DB, FC: and because DF is little in comparison of DA, therefore [20.] DF will be to HB nearly as five to one; therefore HB will be little in comparison of AB; and therefore the rectangles AHB, ABH, will be nearly equal; therefore, because the mean distance of the moon from the earth is to one third part of

the distance of the sun from the earth in the subduplicate ratio of the rectangle AHB to the square of AC, and the rectangle AHB is nearly equal to the rectangle ABH, and the rectangle ABE is equal to the square of AC; therefore the mean distance of the moon from the earth will be to one third part of the distance of the sun from the earth nearly in the subduplicate ratio of the rectangle ABH to the rectangle ABE: but the rectangle ABH is to the rectangle ABE as BH to BE; that is, (because DF is nearly quintuple of BH and DE is quintuple of BE) as DF to DE; therefore the mean distance of the moon from the earth will be to one third part of the distance of the sun from the earth nearly in the subduplicate ratio of DF to DE. Q. E. D.

P R O P. XXIV. Fig. 108.

Let AB be any given line bisected in C; take the line p, such, that CB may be to p in the duplicate ratio of the periodic time of the earth round the sun to the periodic time of the moon round the earth; and in AB take

Y y 2 BD,

BD, such, that BD may be to p as the force of the moon to the earth to the mean disturbing force of the sun upon the moon; in CB produced towards B take BE, such, that CB may be double of BE; upon AE let there be a semicircle AFE described; draw DF perpendicular to AE, meeting the semicircle in F; and join AF: the mean distance of the moon from the earth will be to the mean distance of the sun from the earth nearly as AF to the triple of AE.

Join FE. Because the angle AFE is an angle in a semicircle, it is therefore a right angle; and because FD is perpendicular to AE, the triangles ADF, AFE, will be similar; therefore AD will be to AF as AF to AE; therefore the square of AF will be to the square of AE as AD to AE; and therefore AF will be to AE in the subduplicate ratio of AD to AE: but the mean distance of the moon from the earth is to one third part of the distance of the sun from the earth [23.] nearly in the subduplicate ratio of AD to AE; therefore the mean distance of the moon from the earth will

will be to one third part of the distance of the sun from the earth nearly as AF to AE; and therefore the mean distance of the moon from the earth will be to the mean distance of the sun from the earth nearly as AF to the triple of AE. Q. E. D.

P R O P. XXV. Fig. 109.

Let there be an ellipse ABCD whose focus is S; suppose a body acted upon by a centripetal force tending to the focus S to describe the ellipse; draw SA, SB, SC, SD, to any points A, B, C, D, in the ellipse; the sum of the central forces that have acted upon the body when describing the parts AB, CD, of the ellipse, will be to one another as the angle ASB to the angle CSD; likewise the sum of the central forces that have acted upon the body during a revolution round the focus S, will be to the sum of the central forces that have acted upon the body when describing the part AB as four right angles to the angle ASB.

Suppose the part AB of the ellipse to be divided

divided into indefinitely little parts, AE, EF, FG, &c. described in equal times; likewise let the part CD of the ellipse be divided into indefinitely little parts, CH, HK, KL, described in equal times; join SE, SF, SG, &c. SH, SK, SL, &c. and let the parts AE, CH, be described in equal times.

Because the body describing the ellipse is acted upon by a centripetal force tending to the focus of the ellipse, the centripetal force will [24. Tr. 1.] be reciprocally as the square of the distance from the focus S; therefore the centripetal force at A will be to the centripetal force at E as the square of SE to the square of SA: but the angular velocity at A is to the angular velocity at E [6. Tr. 1.] as the square of SE to the square of SA; therefore the centripetal force at A will be to the centripetal force at E as the angular velocity at A to the angular velocity at E: but because the parts AE, EF, are indefinitely little, and described in equal times, the angular velocity at A will be to the angular velocity at E as the angle ASE to the angle ESF; therefore the centripetal

tripetal force at A is to the centripetal force at E as the angle ASE to the angle ESF. The same way it is shown, that the centripetal force at A is to the centripetal force at F as the angle ASE to the angle FSG, and so on; therefore it is evident, that the sum of the central forces that have acted upon the body when describing the part AB is to the centripetal force at A as the angle ASB to the angle ASE. The same way it is shown, that the sum of the central forces that have acted upon the body when describing the part CD is to the centripetal force at C as the angle CSD to the angle CSH; therefore the centripetal force at C will be to the sum of the central forces that have acted upon the body when describing the part CD as the angle CSH to the angle CSD. Because the sum of the central forces that have acted upon the body when describing the part AB is to the centripetal force at A as the angle ASB to the angle ASE, and the centripetal force at A is to the centripetal force at C as the square of SC to the square of SA; that is, as the angle ASE to the angle CSH; therefore the sum of the central

central forces that have acted upon the body when describing the part AB will be to the centripetal force at C as the angle ASB to the angle CSH: but the centripetal force at C is to the sum of the central forces that have acted upon the body when describing the part CD as the angle CSH to the angle CSD; therefore, by equality, the sum of the central forces that have acted upon the body when describing the part AB, will be to the sum of the central forces that have acted upon the body when describing the part CD, as the angle ASB to the angle CSD.

The same way it is shown, that the sum of the central forces that have acted upon the body during a revolution round the focus S, will be to the sum of the central forces that have acted upon the body when describing the part AB, as four right angles to the angle ASB. *Q. E. D.*

P R O P. XXVI. *Fig. 110.*

Let there be two equal and similar ellipses ABCD, abcd, described by two bodies acted

Plate. 17.

Fig. 103

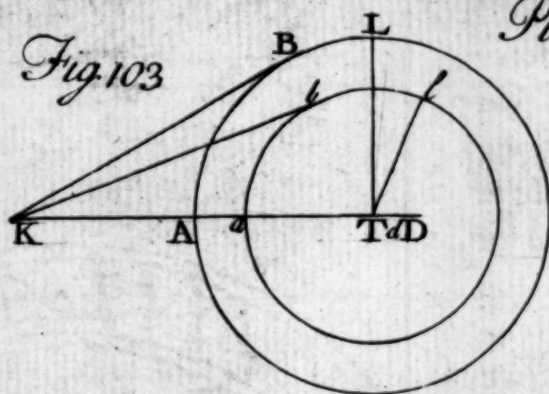


Fig. 104

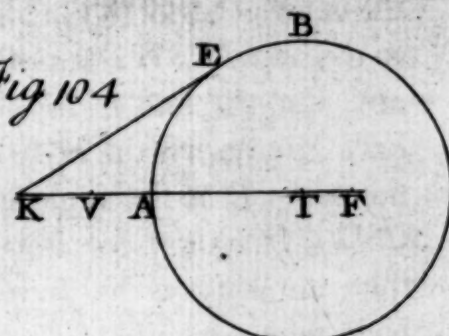


Fig. 106

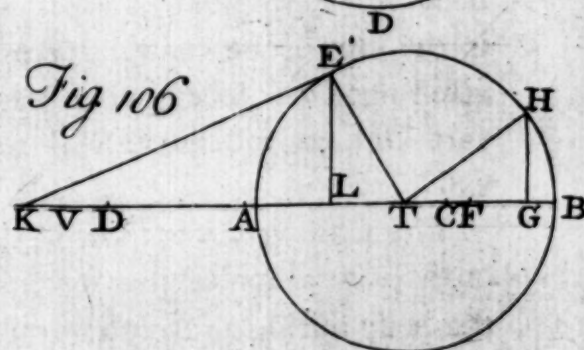


Fig. 105

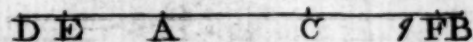


Fig. 109

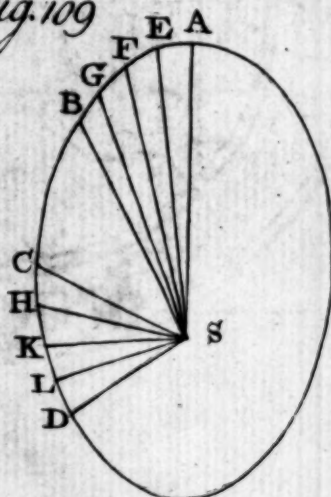


Fig. 107

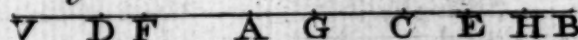
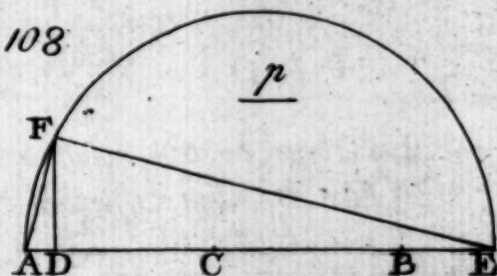


Fig. 108



ted upon by centripetal forces tending to the
 foci E, e ; the sum of the central forces that
 have acted upon the body describing the ellipse
 $ABCD$ during a revolution round the focus
 E , will be to the sum of the central forces that
 have acted upon the body describing the el-
 lipse $abcd$ during a revolution round the fo-
 cus e , as the periodic time of the body descri-
 bing the ellipse $abcd$ to the periodic time of
 the body describing the ellipse $ABCD$.

Let AC be the transverse axis of the el-
 lipse $ABCD$, and ac the transverse axis of
 the ellipse $abcd$; let EA, ea , be equal; and
 let AF be a part of the ellipse $ABCD$ descri-
 bed in an indefinitely little part of time;
 let af be a part of the ellipse $abcd$ described
 in the same time the part AF was described;
 draw FG, fg , perpendicular to EA, ea ,
 meeting EA, ea , in G, g ; join EF, ef ;
 in FG take GH, go , that the rectangle
 FGH may be equal to the square of fg .

Because the ellipses $ABCD, abcd$, are e-
 qual and similar, the square of FG will be
 to the rectangle AGC as the square of fg to
 the rectangle agc ; therefore the square of

Zz

FG

FG will be to the square of fg as the rectangle AGC to the rectangle agc : but because AC, ac , are equal, and AG, ag , are indefinitely little, therefore CG, cg , may be considered as equal; therefore the rectangle AGC will be to the rectangle agc as AG to ag ; therefore the square of FG will be to the square of fg as AG to ag ; that is, the square of FG will be to the rectangle FGH as AG to ag ; therefore FG is to GH as AG to ag ; that is, AG is to ag as FG to GH: but because the angles FEG, GEH, are indefinitely little, the angle FEG is to the angle GEH as FG to GH; therefore AG is to ag as the angle FEG to the angle GEH. Because the parts AF, af , are described in the same time, the central force that has acted upon the body describing the part AF is to the central force that has acted upon the body describing the part af as AG to ag ; therefore the central force that has acted upon the body describing the part AF is to the central force that has acted upon the body describing the part af as the angle FEG to the angle GEH. Again, because the sum of the central forces that

have

have acted upon the body describing the ellipse ABCD during a revolution round the focus E is to the central force that has acted upon the body describing the part AF as four right angles to the angle FEG, and the central force that has acted upon the body describing the part AF is to the central force that has acted upon the body describing the part *af* as the angle FEG to the angle GEH; therefore the sum of the central forces that have acted upon the body describing the ellipse ABCD is to the central force that has acted upon the body describing the part *af* as four right angles to the angle GEH: but the central force that has acted upon the body describing the part *af* is to the sum of the central forces that have acted upon the body describing the ellipse *abcd* during a revolution round the focus *e* as the angle *feg* to four right angles; therefore the sum of the central forces that have acted upon the body describing the ellipse ABCD during a revolution round the focus E, is to the sum of the central forces that have acted upon the body describing the ellipse *abcd* during a revolution round

Z z 2

the

the focus e , as the angle feg to the angle GEH : but because EG , eg , may be considered as equal, and the angles feg , GEH , are indefinitely little, the angle feg is to the angle GEH as fg to GH ; therefore the sum of the central forces that have acted upon the body describing the ellipse $ABCD$ during a revolution round the focus E , is to the sum of the central forces that have acted upon the body describing the ellipse $abcd$ during a revolution round the focus e , as fg to GH . Again, because the periodic time of the body describing the ellipse $ABCD$ is to the time in which the part AF or af is described as the ellipse $ABCD$ or $abcd$ to the sector AEF , and the time in which the part af is described is to the periodic time of the body describing the ellipse $abcd$ as the sector aef to the ellipse $abcd$; therefore the periodic time of the body describing the ellipse $ABCD$ is to the periodic time of the body describing the ellipse $abcd$ as the sector aef to the sector AEF : but because ea , EA , are equal, and the sectors aef , AEF , are indefinitely little, the sector aef is to the sector AEF as fg to FG ; therefore the periodic
time

time of the body describing the ellipse ABCD is to the periodic time of the body describing the ellipse *abcd* as *fg* to FG; therefore, inversely, the periodic time of the body describing the ellipse *abcd* is to the periodic time of the body describing the ellipse ABCD as FG to *fg*; that is, as *fg* to GH; therefore the sum of the forces that have acted upon the body describing the ellipse ABCD during a revolution round the focus E, is to the sum of the forces that have acted upon the body describing the ellipse *abcd* during a revolution round the focus *e*, as the periodic time of the body describing the ellipse *abcd* to the periodic time of the body describing the ellipse ABCD.

Q. E. D.

COR. Let there be two equal circles described by two bodies acted upon by centripetal forces tending to the centres of the circles, and let the centripetal forces be reciprocally as the square of the distance from the centres; the sum of the central forces that have acted upon the body describing the one during a revolution round the centre, will

will be to the sum of the central forces that have acted upon the body describing the other during a revolution round the centre, reciprocally as the periodic times.

P R O P. XXVII. *Fig. III.*

Supposing the original orbit of the moon round the earth to be a circle, if the action of the sun did not affect the moon; but the action of the sun diminishing the gravity of the moon to the earth, the orbit is thereby lengthened: it is required to determine in this case the excentricity of the lunar orbit.

It was shown [10.], that the action of the sun upon the moon in some parts of the lunar orbit diminishes the gravity of the moon to the earth, and in other parts of the lunar orbit increases the gravity of the moon to the earth; but the sum of the forces diminishing the gravity of the moon to the earth is greater than the sum of the forces increasing the gravity of the moon to the earth; and therefore, upon the whole, the action of the sun upon the moon diminishes

nishes the gravity of the moon to the earth. Suppose the original orbit of the moon round the earth to be a circle, if the action of the sun did not affect the moon; but the action of the sun diminishing the gravity of the moon to the earth, the orbit of the moon is thereby lengthened, and the lunar apogee has a progressive motion.

Let TA represent the distance of the moon from the earth at T; produce TA to B, and let AB be equal to AT; draw AC perpendicular to AT, and let AC be equal to AT; let CSD be a curve, such, that if from any point M in AB there be drawn MS perpendicular to AB, meeting the curve CSD in S, the square of TM is to the square of TA as AC to MS; draw BD perpendicular to AB, meeting the curve CSD in D; let AC represent the centripetal force of the moon to the earth at the distance TA, supposing the action of the sun not to affect the moon; MS, BD, likewise will represent the centripetal forces of the moon to the earth at the distances TM, TB, supposing the action of the sun not to affect the moon; in AC take AE, such,
that

that AC may be to AE as the centripetal force of the moon to the earth at the distance TA to the mean disturbing force of the sun upon the moon at the same distance; join TE meeting MS in F; and because the mean disturbing forces of the sun upon the moon at different distances of the moon from the earth are to one another [18.] nearly as the distances of the moon from the earth; therefore, if AE represent the mean disturbing force of the sun upon the moon at the distance TA, MF will represent the mean disturbing force of the sun upon the moon at the distance TM. Let AHK represent the orbit described by the moon round the earth by the joint forces of the earth and sun; let K be any point in the orbit, and let KL be a part of the orbit described in an indefinitely little part of time; join TK, TL; with the centre T, and distances TK, TL, describe the circles KM, LN, meeting TB in M, N; let the circle LN meet TK in O; let AH be a part of the orbit AHK described in a time equal to the time in which the part LK was described; and join TH: because the original orbit of the
moon

moon round the earth was supposed to be a circle, the velocity at the point A of the moon in its orbit will be equal [3. Tr. 3.] to the velocity that would be acquired by falling from B to A; and the space AD will represent the sum of the forces that have acted upon the moon in its descent from B to A; but at the point A, the action of the sun upon the moon diminishing the gravity of the moon to the earth, the moon will recede from the centre, and the orbit will be lengthened; and because AC represents the centripetal force of the moon to the earth at the distance TA, and AE the mean disturbing force of the sun upon the moon at the same distance diminishing the gravity of the moon to the earth; therefore EC will represent the force of the moon to the earth at the distance TA. The same way it is shown, that FS will represent the force of the moon to the earth at the distance TM; and the space ES will represent the sum of the forces that have acted upon the moon in its ascent through the space AM; therefore the velocity of the moon at the distance

3 A

TM

TM will be to the velocity of the moon at the distance TA in the subduplicate ratio of the sum of the spaces DM, AF, to the space AD: and because the parts LK, AH, of the orbit are described in equal times, therefore LK will be to AH as the velocity at K to the velocity at A; that is, as the velocity at M to the velocity at A; therefore LK will be to AH in the subduplicate ratio of the sum of the spaces DM, AF, to the space AD; therefore the square of LK will be to the square of AH as the sum of the spaces DM, AF, to the space AD.

With the centre T, and distance TA, describe a circle meeting TA in Z; in AM produced take Mf quadruple of MA; and take Av, such, that MA may be to Av as TA to AE.

Because the square of TM is to the square of TA as the triangle TMF to the triangle TAE; that is, as twice the triangle TMF to twice the triangle TAE; by division, the rectangle ZMA will be to the square of TA as twice the space AF to the rectangle TAE; and alternating, the rectangle

angle

angle ZMA will be to twice the space AF as the square of TA to the rectangle TAE; that is, as TA to AE; that is, as AM to Av; that is, as the rectangle ZMA to the rectangle contained by ZM, Av; therefore the rectangle contained by ZM, Av, is double of the space AF: because the mean disturbing force of the sun upon the moon is little in comparison of the centripetal force of the moon to the earth; therefore AM will be little in comparison of TA; and therefore the squares of ZM, ZA, may be considered as equal; therefore the square of ZM may be considered as quadruple the square of TA; therefore ZM will be to TA as four times TA to ZM: but the rectangle ZMA is to the rectangle TAM as ZM to TA; therefore the rectangle ZMA will be to the rectangle TAM as four times TA to ZM: but the rectangle TAM is to the square of TA as AM to TA; that is, as four times AM, that is, Mf, to four times TA; therefore, by perturbate equality, the rectangle ZMA will be to the square of TA as Mf to ZM; therefore, by composition,

as $3A^2$ to $2A^2$ so the

the square of TM will be to the square of TA as Zf to ZM; that is, as the rectangle contained by Zf, Av, to the rectangle contained by ZM, Av: but the rectangle contained by ZM, Av, is double of the space AF; therefore the square of TM will be to the square of TA as the rectangle contained by Zf, Av, to twice the space AF. Again, the square of TM is to the rectangle TMB as TM to MB; that is, [11. Tr. 2.] as the rectangle TBD to the space DM; that is, as twice the rectangle TBD to twice the space DM; and twice the rectangle TBD is equal to the rectangle TAC; that is, equal to the square of TA; therefore the square of TM will be to the rectangle TMB as the square of TA to twice the space DM; and alternating, the square of TM will be to the square of TA as the rectangle TMB to twice the space DM: but it has been shown, that the square of TM is to the square of TA as the rectangle contained by Zf, Av, to twice the space AF; therefore the square of TM will be to the square of TA as the rectangle TMB together with the rectangle contained by Zf, Av, to twice the
the

the sum of the spaces DM , AF : but because AH , LK , are described in equal times, the triangles TAH , TKL , will be equal; therefore AH will be to LO as TK or TM to TA ; therefore the square of AH will be to the square of LO as the square of TM to the square of TA ; therefore the square of AH will be to the square of LO as the rectangle TMB , together with the rectangle contained by Zf , Av , to twice the sum of the spaces DM , AF : but because the square of LK is to the square of AH as the sum of the spaces DM , AF , to the space AD ; that is, as twice the sum of the spaces DM , AF , to twice the space AD , and the square of AH is to the square of LO as the rectangle TMB , together with the rectangle contained by Zf , Av , to twice the sum of the spaces DM , AF ; therefore, by perturbate equality, the square of LK will be to the square of LO as the rectangle TMB , together with the rectangle contained by Zf , Av , to twice the space AD : but because TA , AB , are equal, twice the space AD is equal to the rectangle TAC ; that is, equal to the square of TA ; therefore the
square

square of LK is to the square of LO as the rectangle TMB , together with the rectangle contained by Zf , Av , to the square of TA .

Let the rectangle TMB , together with the rectangle contained by Zf , Av , be equal to the space Q , together with the square of TA ; in fT take fq , such, that fq may be to AM as TA to AE ; and in AT take AG quintuple of AE ; in AB take AP , such, that TG may be to TA as twice AE to AP ; upon AP let there be a semicircle described; draw MQ ; Nr , perpendicular to AP , meeting the semicircle in Q , r ; and draw rs parallel to AP , meeting MQ , in S ; bisect AP in x ; and join Qx ; in Tx take xp , such, that the rectangle Txp may be equal to the square of xA ; draw pt perpendicular to Qt , a tangent to the semicircle at Q ; and join pQ , pr ; let the square of TA be to the square of Ay as TG to TA ; in GA take Gd , da , equal to AE ; and in AG take Am equal to AE .

Because the rectangle TMB , together with the rectangle contained by Zf , Av , is equal to the space Q , together with the square of TA ; let the square of AM be add-

ed

ed to both, and the square of TA , together with the rectangle contained by Zf , Av , will be equal to the space Q , together with the sum of the squares of TA , AM ; therefore taking the square of TA from both, the rectangle contained by Zf , Av , will be equal to the space Q , together with the square of AM : but because fq is to AM as TA to AE ; that is, as AM to Av , the square of AM will be equal to the rectangle contained by fq , Av ; therefore the rectangle contained by Zf , Av , will be equal to the space Q , together with the rectangle contained by fq , Av ; therefore taking the rectangle contained by fq , Av , from both, the rectangle contained by Zq , Av , will be equal to the space Q . Again, because TA is to AE as fq to AM , and AE is to AG as AM to Af , TA will be to AG as fq to Af ; therefore, by division, TG will be to GA as Aq to Af : but GA is to AE as Af to AM ; therefore TG is to AE as Aq to AM : but because TG is to TA as twice AE to AP ; and TA is to AZ as AP to twice AP ; therefore, by equality, TG will be to AZ as twice AE to twice AP ; that is,

is, as AE to AP; and alternating, TG will be to AE as AZ to AP: but TG is to AE as Aq to AM; therefore AZ is to AP as Aq to AM; therefore AZ will be to AP as Zq to PM; that is, as the rectangle contained by Zq, Av, to the rectangle contained by PM, Av: but the space Q is equal to the rectangle contained by Zq, Av; therefore AZ will be to AP as the space Q to the rectangle contained by PM, Av; that is, the space Q will be to the rectangle contained by PM, Av, as AZ to AP: but the rectangle contained by PM, Av, is to the rectangle PMA as Av to AM; that is, as AE to AT; that is, as twice AE to AZ; therefore, by perturbate equality, the space Q will be to the rectangle PMA, that is, to the square of MQ, as twice AE to AP; that is, as TG to TA; that is, as the square of TA to the square of Ay; therefore, alternating, the space Q will be to the square of TA as the square of MQ to the square of Ay.

Again, because it has been shown, that the square of LK is to the square of LO as the rectangle TMB, together with the rectangle contained by Zf, Av, to the square of

of TA, and the rectangle TMB, together with the rectangle contained by Zf, Av, is equal to the space Q, together with the square of TA; therefore the square of LK will be to the square of LO as the space Q, together with the square of TA, to the square of TA; therefore, by division, the square of KO will be to the square of LO as the space Q to the square of TA; that is, as the square of MQ to the square of Ay; therefore KO will be to LO as MQ to Ay; that is, MN will be to LO as MQ to Ay; therefore LO will be to MN as Ay to MQ: but because Qr is indefinitely little, the triangles Qrs, QxM, will be similar; and rs, that is, MN, will be to Qr as MQ to Qx; therefore LO will be to Qr as Ay to Qx or Ax; therefore the rectangle contained by TM, LO, will be to the rectangle contained by TM, Qr, as Ay to Ax: but the rectangle contained by TM, Qr, is to the rectangle contained by pt, Qr, as TM to pt; that is, [4. Tr. 2.] as Ax to xp; therefore the rectangle contained by TM, LO, is to the rectangle contained by pt, Qr, as Ay to px: but the rectangle con-

tained by TM , LO , that is, the rectangle contained by TK , LO , is double of the sector TKL ; and the rectangle contained by pt , Qr , is double of the sector pQr ; therefore the sector TKL is to the sector pQr as Ay to px .

Supposing the space ATK to be divided into indefinitely little sectors, and repeating the construction, the sector ApQ of the semicircle will likewise be divided into indefinitely little sectors, and each of the sectors in the space ATK will be to its corresponding sector in the semicircle as Ay to px ; therefore it is evident, that the space ATK will be to the sector ApQ as Ay to px ; and that the space described round the centre T , while the moon recedes through the distance AP from the centre, will be to the semicircle AQP as Ay to px ; that is, as the rectangle contained by Ay , Tx , to the rectangle Txp : but the semicircle AQP is to the semicircle AHZ as the square of Ax to the square of TA ; that is, as the rectangle Txp to the square of TA ; therefore the space described round the centre T , while the moon recedes from the centre T through the

the space AP, will be to the semicircle AHZ as the rectangle contained by Ay, Tx, to the square of TA; and therefore it is evident, that the space described by the moon round the earth from apogee to apogee will be to the circle whose semidiameter is TA as the rectangle contained by Ay, Tx, to the square of TA; therefore the time the moon takes to revolve from apogee to apogee, will be to the time the moon would describe a circle round the earth at the distance TA by the action of the earth alone, as the rectangle contained by Ay, Tx, to the square of TA.

Because TG is to Ta as TA to TP, the square of TG is to the square of Ta as the square of TA to the square of TP; and because Gm is bisected in a, and Tm, TG, are nearly equal; therefore TG, Ta, Tm, may be considered as proportionals; therefore TG will be to Tm as the square of TG to the square of Ta; that is, as the square of TA to the square of TP; therefore the parallelopiped whose base is the square of TP, and altitude TG, will be equal to the parallelopiped whose base is the square of TA, and altitude Tm.

3 B 2

Again,

Again, because the square of Ay is to the square of TA as TA to TG ; therefore Ay will be to TA in the subduplicate ratio of TA to TG ; therefore the rectangle contained by Ay , Tx , will be to the rectangle contained by TA , Tx , in the subduplicate ratio of TA to TG : and because the rectangle contained by TA , Tx , is to the square of TA as Tx to TA ; that is, in the subduplicate ratio of TP to TA , (because TP , Tx , TA , may be considered as proportionals); therefore the rectangle contained by Ay , Tx , will be to the square of TA in the subduplicate ratio of TP to TG ; that is, in the subduplicate ratio of the cube of TP to the paralleliped whose base is the square of TP , and altitude TG ; that is, in the subduplicate ratio of the cube of TP to the paralleliped whose base is the square of TA , and altitude Tm : but the time the moon takes to revolve from apogee to apogee, is to the time in which the moon would describe a circle round the earth at the distance TA by the action of the earth alone, as the rectangle contained by Ay , Tx , to the square of TA ; therefore the time the moon takes to revolve from
apogee

apogee to apogee, will be to the time in which the moon would describe a circle round the earth at the distance TA by the action of the earth alone, in the subduplicate ratio of the cube of TP to the parallelopiped whose base is the square of TA, and altitude Tm : but the time in which the moon would describe a circle round the earth at the distance TA by the action of the earth alone, is to the time in which the moon revolves round the earth by the joint forces of the earth and sun, in the subduplicate ratio of CE to CA; that is, in the subduplicate ratio of Tm to TA; that is, in the subduplicate ratio of the parallelopiped whose base is the square of TA, and altitude Tm , to the cube of TA; therefore the time the moon takes to revolve from apogee to apogee is to the periodic time of the moon round the earth, in the subduplicate ratio of the cube of TP to the cube of TA; therefore the cube of TP is to the cube of TA, in the duplicate ratio of the time the moon takes to revolve from apogee to apogee to the periodic time of the moon round the earth. Q. E. I.

PROP.

PROP. XXVIII. Fig. 112. 113. 114. 115.

Let there be an ellipse whose lesser axis is AB , and centre C ; let CD be half the greater axis; in AC take the point E , such, that the square of AC is to the rectangle AEB as the square of DC to the square of CA ; bisect EC in F , and in CB take the point G , such, that GF may be to FC as the square of DC to the square of CA ; draw FH perpendicular to CF , meeting the semiellipse AHD in the point H ; and join GH : of all the lines that can be drawn from the point G to any point in the semiellipse, the line GH is the greatest.

In FG produced take GK equal to FG .

1. Fig. 112. From the point G let there be drawn GD to the point D ; from H draw HL parallel to AB , meeting CD in L ; and let CD meet the ellipse in N .

Because KC is greater than GF , the ratio of KC to FC will be greater than the ratio of GF to FC ; therefore the ratio of
the

the rectangle KCF to the square of FC will be greater than the ratio of GF to FC: but GF is to FC as the square of DC to the square of CA; that is, because of the ellipse, as the rectangle DLN to the square of HL; therefore the ratio of the rectangle KCF to the square of FC is greater than the ratio of the rectangle DLN to the square of HL: and because the squares of FC, HL, are equal, because FC, HL, are equal, therefore the rectangle KCF is greater than the rectangle DLN: let the square of CG be added to both, and the square of GF will be greater than the rectangle DLN together with the square of CG. Again, let the square of HF or CL be added to both, and the squares of GF, FH, will be greater than the squares of GC, CD: but the square of GH is equal to the squares of GF, FH, and the square of GD is equal to the squares of GC, CD; therefore the square of GH will be greater than the square of GD; and therefore GH is greater than GD.

2. *Fig. 113.* Again, from the point G, let there be drawn GO to any point O; between

tween A, H, draw OP perpendicular to AB, meeting AB in P; let HF meet the ellipse in Q; and draw OR parallel to AB, meeting HF, CD, in R, S, and the ellipse in T.

Because the ratio of PC to CF is greater than the ratio of PG to GF; by composition, the ratio of PC together with CF to CF will be greater than the ratio of PK to GF: but PC together with CF is equal to OS together with SR; that is, equal to RT; therefore the ratio of RT to CF is greater than the ratio of PK to GF; therefore, alternating, the ratio of RT to PK is greater than the ratio of CF to FG; therefore the ratio of the rectangle ORT to the rectangle KPF is greater than the ratio of CF to FG: but CF is to FG as the square of AC to the square of CD; that is, because of the ellipse, as the rectangle ORT to the rectangle HRQ; therefore the ratio of the rectangle ORT to the rectangle KPF is greater than the ratio of the rectangle ORT to the rectangle HRQ; and therefore the rectangle HRQ is greater than the rectangle KPF: let the square of GF be added to both, and the

the rectangle HRQ together with the square of GF will be greater than the square of GP . Let the square of FR or OP be added to both, and the squares of GF , FH , will be greater than the squares of GP , PO : but the square of GH is equal to the squares of GF , FH , and the square of GO is equal to the squares of GP , PO ; therefore the square of GH is greater than the square of GO ; and therefore GH is greater than GO .

3. *Fig. 114.* Again, from the point G let there be drawn GO to any point O in the ellipse between D , H ; draw OP perpendicular to AB , meeting AB in P , and the ellipse in Q ; and draw HR parallel to AB , meeting OP , CD , in R , S , and the ellipse in T .

Because the ratio of GP to PC is greater than the ratio of GF to FC ; therefore, alternating, the ratio of GP to GF will be greater than the ratio of PC to CF ; therefore the ratio of GP together with GF to GF , that is, the ratio of KP to GF , will be greater than the ratio of PC together with

3 C

CF

CF to CF : but FC together with CP is equal to HS together with SR ; that is, equal to RT ; therefore the ratio of KP to GF will be greater than the ratio of RT to FC ; therefore, alternating, the ratio of KP to RT is greater than the ratio of GF to FC ; and therefore the ratio of the rectangle KPF to the rectangle HRT is greater than the ratio of GF to FC : but GF is to FC as the square of DC to the square of CA ; that is, because of the ellipse, as the rectangle ORQ to the rectangle HRT ; therefore the ratio of the rectangle KPF to the rectangle HRT is greater than the ratio of the rectangle ORQ to the rectangle HRT ; therefore the rectangle KPF is greater than the rectangle ORQ. Let the square of GP be added to both, and the square of GF will be greater than the rectangle ORQ together with the square of GP. Again, let the square of HF or PR be added to both, and the squares of GF, FH, together, will be greater than the squares of GP, PO, together : but the square of GH is equal to the squares of GF, FH, together, and the square of GO is equal to the squares of GP, PO,

PO, together; therefore the square of GH is greater than the square of GO; and therefore GH is greater than GO.

4. *Fig. 115.* Again, from the point G let there be drawn GO to any point O in the ellipse between D, B; draw OP parallel to AB, meeting CD in P, and the ellipse in Q; and join CO, GQ, CQ.

Because OQ is parallel to AB, OP, PQ, will be equal, and the angles at P right; therefore CO, CQ, are equal; therefore the angles CQO, COQ, are equal: but the angle CQO is greater than the angle GQO; therefore the angle COQ is greater than the angle GQO: but the angle GOQ is greater than the angle COQ; therefore the angle GOQ is greater than the angle GQO; therefore GQ is greater than GO; and therefore GH is greater than GO.

2. E. D.

COR. From this it is evident, that a circle described with the centre G, and distance H, will fall without the ellipse ADB.

P R O P. XXIX. *Fig. 116.*

Let there be a semiellipse whose lesser axis is AB , and centre C ; and let CD be half the greater axis; in AC take the point E , such, that the square of AC is to the rectangle AEB as the square of CD to the square of CA ; and let EC be to CF as the square of AC together with the rectangle AEB to the square of EC ; from C, E , let there be drawn CG, EG , to any point G in the semiellipse, meeting the semicircle described upon AB in H, K : the sine of the arc HK will be less than CF .

Bisect EC in L ; draw LM perpendicular to EL , meeting the ellipse in M ; and in EC produced take the point N , such, that EN is to NC as EC to CF ; join MN, CM ; with the centre N , and distance NM , let there be a circle described, meeting AB in O, P ; let EG meet the circle OMP in Q , and join NQ ; let LM meet the circle AHB in R , and CM meet the circle AHB in Z ; and let MS a tangent to the ellipse at M meet

meet AB in S; join SR, CR; from H draw HT parallel to AB, meeting CG in T; join EM, and let EM meet the circle AHB in V; draw VX parallel to AB, meeting CM in X; and in CS take EY equal to EC.

Because EN is to NC as EC to CF; that is, as the rectangle AEB together with the square of AC to the square of EC; by division, EC will be to CN as twice the rectangle AEB to the square of EC; therefore LC will be to CN as the rectangle AEB to the square of EC; therefore LC will be to LN as the rectangle AEB to the square of AC; and therefore, inversely, NL will be to LC as the square of AC to the rectangle AEB: but the square of LM is to the square of LR as the square of CD to the square of CA; that is, as the square of AC to the rectangle AEB; and the rectangle NLS is to the rectangle CLS as NL to LC; therefore the rectangle NLS is to the rectangle CLS as the square of LM to the square of LR: but because SM is a tangent to the ellipse at M, SR will be a tangent to the circle at R; therefore the angle SRC is right; and

and therefore the rectangle CLS is equal to the square of LR; therefore the rectangle NLS is equal to the square of LM; and therefore the angle SMN is a right angle. Again, because LC is to CN as the rectangle AEB to the square of EC; by composition, LN will be to NC as the square of AC to the square of EC: but the rectangle SCL is equal to the square of CR or CA; and the rectangle YCL is equal to the square of CE, because YC is double of CE, and CE is double of CL; therefore LN will be to NC as the rectangle SCL to the rectangle YCL; that is, as SC to CY; and therefore, by division, LC will be to CN as SY to YC; therefore the rectangle contained by SY, CN, will be equal to the rectangle YCL; that is, equal to the square of YE or EC; therefore SY will be to YE as YE or EC to CN; and therefore, by composition, SE will be to EC as EN to NC; therefore alternating, SE will be to EN as EC to CN; therefore, by composition, SN will be to NE, as EN to NC; therefore the rectangle SNC is equal to the square of EN. Because the rectangle

SNL

SNL is to the rectangle SNC as LN to NC; that is, as the square of AC to the square of EC; and the rectangle SNL is equal to the square of NM, because the angle SMN is right, and the rectangle SNC was shown to be equal to the square of EN; therefore the square of NM will be to the square of EN as the square of AC or CV to the square of CE; therefore NM will be to EN as VC to CE: and because the angle CEV is common to both the triangles VEC, MEN, the triangles VEC, MEN, will be equiangular, and the angles ECV, ENM, will be equal; therefore CV, MN, are parallel. The same way it is shown, that CH, QN, are parallel; therefore EM is to MV as EN to NC; that is, as EC to CF: but EC is to VX as EM to MV; therefore EC is to VX as EC to CF; therefore VX is equal to CF: but because the angle VXM is equal to the angle ECM, the angle VXM will be less than a right angle; therefore VX will be greater than the sine of the arc VZ; and therefore the sine of the arc VZ is less than CF: and because [cor. prop. 28.] the circle described with the centre N, and distance NM, falls without

without the ellipse ADB, therefore the point Q falls without the ellipse; therefore the ratio of EG to GH is greater than the ratio of EQ to QH: but EC is to HT as EG to GH, and EC is to CF as EN to NC; that is, as EQ to QH; therefore the ratio of EC to HT is greater than the ratio of EC to CF; therefore HT is less than FC: but the sine of the arc HK is not greater than HT; and therefore the sine of the arc HK is less than CF. Q. E. D.

P R O P. XXX. Fig. 117.

Let there be a circle whose diameter is AB, and centre C, and let D be a point in the diameter; in CD take CE, such, that the rectangle ACE be equal to the square of CD; take CV, such, that DC is to CV as the sum of CA and AE is to CE; from D let there be drawn DF to any point F in the circle; and from C let there be drawn CG parallel to DF, meeting FG a tangent to the circle at F in G, and likewise meeting the semicircle AFB in H; join DG meeting the semicircle AFB in K: the sine of the arc HK is less than CV.

Because

Because the rectangle ACE is equal to the square of CD, the rectangle CAE will be equal to the rectangle ADB; therefore the rectangle ADB together with the square of AC will be equal to the rectangle CAE together with the square of AC: but the rectangle CAE together with the square of AC is equal to the rectangle contained by CA, and the sum of CA, AE; therefore the rectangle ADB together with the square of AC is equal to the rectangle contained by CA, and the sum of CA, AE; and therefore the rectangle ADB together with the square of AC will be to the square of CD as the rectangle contained by CA, and the sum of CA, AE, to the rectangle ACE: but the rectangle contained by CA, and the sum of CA, AE, is to the rectangle ACE as the sum of CA, AE, to CE; that is, as DC to CV; therefore the rectangle ADB together with the square of AC is to the square of CD as CD to CV.

Again, draw CL perpendicular to AC, and let the square of CL be to the square of CA as the square of CA to the rectangle

ADB;

ADB; draw GM perpendicular to AB , meeting AB in M , and the circle in N , O ; and join CF . Because GF is a tangent to the circle at F , the angle GFC is a right angle; therefore the rectangle GFC is double of the triangle GFC : and because GM is perpendicular to AB , the rectangle contained by DC , GM , is double of the triangle GDC : but because DF , GC , are parallel, the triangles GFC , GDC , are equal; therefore the rectangle GFC is equal to the rectangle contained by DC , GM ; therefore GM is to GF as CF to CD ; therefore the square of GM is to the square of GF , or the rectangle NGO , as the square of CF or CA to the square of CD ; therefore, conversely, the square of GM is to the square of MN as the square of CA to the rectangle ADB ; that is, as the square of CL to the square of CA ; therefore GM is to MN as CL to CA ; and therefore it is evident, that the point G is in an ellipse described upon the femiaxes CL , CA ; and therefore (by the last proposition) the sine of the arc HK is less than CV . Q. E. D.

P R O P.

P R O P. XXXI. . Fig. 118.

Let there be a straight line AB, and CD a portion of a curve wholly concave towards AB, and draw AC, BD, parallel to each other, meeting the curve in C, D; let CE, a tangent to the curve at C, meet BD in E; join AD, AE, BC; and let AE meet the curve in F: the triangle ABC will be greater than the sector ACF, but less than the sector ACD.

Join CD. Because AC, BE, are parallel, the triangles ABC, AEC, will be equal: but the triangle AEC is greater than the sector ACF; therefore the triangle ABC is greater than the sector ACF. Again, because AC, BD, are parallel, the triangles ABC, ADC, are equal: but the triangle ADC is less than the sector ACD; therefore the triangle ABC is less than the sector ACD.

3 D 2

P R O P.

PROP. XXXII. *Fig. 119.*

Let there be a curve AEB, wholly concave towards the straight line AB, and let C, D, be two points in the line AB; draw DE to any point E in the curve, and draw CF parallel to DE, meeting the curve in F; let EG, a tangent to the curve at E, meet CF in G, and join DF, DG, and let DG meet the curve in H; let the point D be between the points A, C: the sector ACE will be greater than the sector ADH, but less than the sector ADF; and the sector BCE will be greater than the sector BDF, but less than the sector BDH.

Because the triangle DEC is [31.] greater than the sector EDH, let the sector ADE be added to both; and the sector ACE will be greater than the sector ADH: and, because the triangle DEC is [31.] less than the sector EDF, let the sector ADE be added to both; and the sector ACE will be less than the sector ADF.

Again, because the sector ACE together with

with the sector BCE, is equal to the sector ADF together with the sector BDF, and the sector ACE is less than the sector ADF; therefore the sector BCE is greater than the sector BDF: and because the sector ACE together with the sector BCE, is equal to the sector ADH together with the sector BDH, and the sector ACE is greater than the sector ADH; therefore the sector BCE is less than the sector BDH. Q. E. D.

P R O P. XXXIII. Fig. 120.

Let there be a semicircle whose diameter is AB, and centre C, and let D be a point in the diameter not very excentric; granting the quadrature of the circle, it is required to draw a line DE, meeting the circle in E, so, that the semicircle may be to the sector BDE in a given ratio, suppose that of p to q .

Suppose the problem solved; and let the semicircle be to the sector BCF as p is to q ; join DF, and draw CG parallel to DF, meeting FG a tangent to the circle at F in G;

G; join DG, and let CG, DG, meet the circle in H, K; join DH, FH; draw CM parallel to DH, meeting HM a tangent to the circle at H in M; draw DL perpendicular to CH, meeting CH in L; join LE, and let FN perpendicular to CH meet CH in N.

Because the semicircle is to the sector BDE as p is to q , that is, as the semicircle to the sector BCF, the sector BDE will be equal to the sector BCF: and because the sector BCH is common to both, the space HCDE will be equal to the sector HCF; but because CH, DF, are parallel, the triangle DCH is equal to the triangle FCH; therefore the sector EDH will be equal to the space contained by the arc HKF, and the chord FH. Again, because the sector BDE is equal to the sector BCF, and the sector BCF is [32.] greater than the sector BDH, and less than the sector BDK, the sector BDE will be greater than the sector BDH, and less than the sector BDK; therefore the point E is in the arc HK between the points H, K. [Let the arc HK be called the limiting arc]. Because the
point

point D is not very excentric, the limiting arc will be little ; therefore EH may be considered as coinciding with the tangent to the circle at the point H ; and therefore the triangle ELH will be equal to the sector EDH ; that is, equal to the space contained by the arc FKH, and the chord FH ; therefore the rectangle LHE will be double of the space contained by the arc FKH, and the chord FH. Because the rectangle contained by CH, and the arc FKH, is double of the sector CFH, and the rectangle contained by CH, FN, is double of the triangle CFH ; the rectangle contained by CH, and the excess of the arc FKH above FN, will be double of the space contained by the arc FKH, and the chord FH ; therefore the rectangle LHE will be equal to the rectangle contained by CH, and the excess of the arc FKH above FN ; therefore LH will be to HC as the excess of the arc FKH above FN to HE : but because the triangles DLH, CHM, are similar, LH will be to HC as DL or FN to HM ; therefore FN will be to HM as the excess of the arc FKH above FN to HE.

From

From this may be deduced the following

CONSTRUCTION.

Let the semicircle be to the sector BCF as p is to q ; join DF; draw CH parallel to DF, meeting the circle in H; join DH; draw CM parallel to DH, meeting HM a tangent to the circle at H in M; and let FN perpendicular to CH meet CH in N; in the tangent HM take HE towards F, so that FN may be to HM as the excess of the arc FKH above FN to HE; join DE; the semicircle will be to the sector BDE nearly as p is to q .

Let DL perpendicular to CH meet CH in L; join EH; LE; let CH meet FG, a tangent to the circle at F, in G; join DG meeting the circle in K.

Because the triangles DLH, CHM, are similar, LH will be to HC as DL or FN to HM; that is, as the excess of the arc FKH above FN to HE; therefore the rectangle LHE will be equal to the rectangle contained by HC, and the excess of the arc HKF above FN; but because the rectangle contained by CH and the arc HF is double

double of the sector HCF, and the rectangle contained by CH, FN, is double of the triangle CFH; therefore the rectangle contained by CH, and the excess of the arc FKH above FN will be double of the space contained by the arc FKH, and the chord FH; therefore the rectangle LHE will be double of the space contained by the arc FKH, and the chord FH; and therefore the triangle LHE will be equal to the space contained by the arc FKH, and the chord FH: but because DL, HE, are parallel, the triangles DHE, LHE, are equal; therefore the triangle DHE is equal to the space contained by the arc FKH, and the chord FH: and because DF, CH, are parallel, the triangles CDH, CFH, are equal; therefore the space HCDE will be equal to the sector CFH. Let the sector BCH be added to both, and the sector BDE will be equal to the sector BCF: but because the sector BCF is greater than the sector BDH, and less than the sector BDK; therefore the sector BDE is greater than the sector BDH, and less than the sector BDK; and therefore DE falls between DH, DK: and be-

cause the point D is not very excentric; the limiting arc HK will be little; therefore the tangent HE may be considered as coinciding with the arc HK: because the sector BDE is equal to the sector BCF, the semicircle will be to the sector BDE as the semicircle to the sector BCF; but the semicircle is to the sector BCF as p is to q ; therefore the semicircle is to the sector BDE nearly as p is to q . Q. E. D.

Kepler first of all discovered that the planets revolved in ellipses round the sun placed in one of the foci, and that they described equal areas in equal times round the sun. Let the semiellipse [*fig. 121.*] whose greater axis is AP, focus S, and centre C, represent half the orbit of a planet round the sun in S; and suppose the planet at the point K in its orbit; join SK: half the periodic time of the planet round the sun is to the time in which the planet moves from A to K as the area of the semiellipse to the area ASK; and therefore, to find the place of the planet at any given time, it is necessary to find the position of the right line SK, which shall

shall cut off the area ASK proportional to the time, that is, to draw the line SK so, that the area of the semiellipse may be to the area ASK as half the periodic time of the planet round the sun to the given time.

From K let fall KH perpendicular to AP, meeting AP in H, and the semicircle described upon AP in G; and join SG: it is evident, from the nature of the ellipse and circle, that the semicircle is to the sector ASG as the semiellipse to the sector ASK; therefore the semicircle is to the sector ASG as half the periodic time of the planet round the sun to the time in which the planet moves from A to K. The problem therefore is reduced to this: To draw the line SG meeting the semicircle in G, so, that the semicircle may be to the sector ASG as half the periodic time of the planet round the sun to the given time. In the semicircle take the arc AB, so, that the arc APB may be to the arc AB as half the periodic time of the planet round the sun to the given time; join CB; the semicircle therefore will be to the sector ACB as half the periodic time of the planet round the sun to the given time;

that is, as the semicircle to the sector ASG ; therefore the sectors ACB, ASG, are equal ; join CG.

The angle ACB is called by Kepler the mean anomaly, the angle ACG the anomaly of the excentric, and the angle ASK the coequate or true anomaly. The problem therefore is reduced to this : The mean anomaly of a planet being given, to find the anomaly of the excentric and the coequate anomaly.

P R O P. XXXIV. Fig. 121.

Let AP be the greater axis of a planet's orbit, S the focus the place of the sun, A the apbelion, P the peribolion ; upon AP let the semicircle ABP be described ; let C be the centre ; and let the angle ACB be the mean anomaly of the planet at any given time : it is required to find the anomaly of the excentric.

Join SB ; and draw CD parallel to SB, meeting the circle in D ; join SD in BD ; take the arc DG so, that the sine of the angle

angle BCD may be to the tangent of the angle CDS as the excess of the arc BD above its sine to the arc DG; and join CG: the angle ACG will be nearly the anomaly of the excentric.

Join SG; draw BE perpendicular to CD, meeting CD in E; and draw CF parallel to SD, meeting DF a tangent to the circle at D in F.

Because BE is the sine of the angle BCD, and DF the tangent of the angle DCF, that is, of the angle SDC, BE will be to DF as the excess of the arc BD above BE to the arc DG; therefore [33.] the sector ASG is equal to the sector ACB; therefore the angle ACG is the anomaly of the excentric.

The computation is as follows. In the triangle BCS, as the sum of the sides BC, CS, is to the difference of the sides BC, CS, so is the tangent of half the angle ACB to the tangent of half the difference of the angles CSB, CBS; therefore the angles CSB, CBS, will be given; that is, the angles ACD, DCB, will be given. Again, in the triangle CSD, the sum of the sides CD, CS,

is

is to the difference of the sides CD, CS, as the tangent of half the angle ACD to the tangent of half the difference of the angles CSD, CDS; therefore the angle CDS will be given. Again, because as the sine of the angle BCD is to the tangent of the angle CDS, so is the excess of the arc BD above its sine to the arc GD; say, As the radius is to the sine of the angle BCD, so is $57^{\circ}.2957795$ &c. the number of degrees in an angle subtended by an arc equal to the radius, to the number of degrees in an angle subtended by an arc equal to the sine of the angle BCD; (let this angle be called A). Again, as the sine of the angle BCD to the tangent of the angle CDS, so is the excess of the angle BCD above the angle A to the angle GCD. Therefore the angle ACG, the anomaly of the excentric, will be given.

○ In orbits of small excentricity, the angle ACD is nearly the anomaly of the excentric; therefore the following rule will give the anomaly of the excentric very nearly.

○ From the logarithmic tangent of half the mean anomaly, subtract the difference of the

the logarithms of the aphelion and perihelion distances; the remainder is the logarithmic tangent of an angle, which call B: to the angle B, add half the mean anomaly; the sum will give very nearly the anomaly of the excentric *.

In the orbits of Mercury and Mars, if the excess of the angle BCD above the angle A be added to the angle ACD, the sum will be nearly the anomaly of the excentric reckoned from the aphelion.

The anomaly of the excentric being found, the coequate or true anomaly will be found by the resolution of the triangle GCS, thus: From the long. tang. of half the anomaly of the excentric, subtract the difference of the logs. of the aphelion and perihelion distances, the remainder will be the log. tang. of an angle; to this angle add half the anomaly of the excentric; let the sum be called the angle C; to the log. tang.

* In the earth's orbit, even in that part where the limiting arc is greatest, the limiting arc will be less than half a second; and in the lunar orbit, when the excentricity is greatest, the limiting arc, even in that part of the orbit where it is greatest, scarcely exceeds half a minute. This may be easily computed from prop. 30.

of the angle C, add half the sum of the logs. of the aphelion and perihelion distances; from this sum subtract the log. of the mean distance: the remainder will be the log. tang. of the coequate or true anomaly.

Let CL be the lesser axis of the planet's orbit. Because, from the nature of the ellipse, the square of CL is equal to the rectangle ASP, the log. of CL will be equal to half the sum of the logs. of AS, SP: and because the tangent of the angle KSH is to the tangent of the angle GSH as HK to HG; that is, from the nature of the ellipse, as LC to CA; therefore, if to the log. tang. of the angle GSH the log. of CL be added, and from the sum the log. of AC be subtracted, the remainder will be the log. tang. of the angle KSH.

Again, the sine of the true anomaly is to the sine of the anomaly of the excentric as the lesser axis of the orbit to the distance of the planet from the sun.

Because CL is to HK as CG to GH, that is, as the radius to the sine of the anomaly of the excentric, and HK is to KS as the sine of the true anomaly to the radius; therefore

therefore CL is to KS as the sine of the true anomaly to the anomaly of the excentric.

The place of a planet in an elliptic orbit (granting the quadrature of the ellipse) may be found at any given time within a small limit by the following proposition.

PROP. XXXV. Fig. 122.

Let the ellipse whose greater axis is AP, foci S, K, and centre C, represent the orbit of a planet round the sun at S; and supposing the periodic time of the planet round the sun to be known, and likewise the time the planet passed through the aphelion A: as the periodic time of the planet round the sun is to the time elapsed since the planet passed through the point A, so let the area of the ellipse be to the sector ACB; join SB, and draw CD parallel to SB on the same side of AP that SB is; and let CD be equal to CA; join SD; let CD, SD, meet the ellipse in E, F: the true place of the planet is between the points E, F; that is, the

3 F planet

planet is passed the point E, but not come to the point F.

Let G be the place of the planet; join SG ; and join BD meeting AP in H ; and join KB ; draw SL parallel to KB , meeting BD in L . Because CD is parallel to SB , CD will be to SB as CH to HS ; therefore twice CD will be to SB as twice CH to HS : but because twice CD is equal to AP , that is, equal to KB together with BS , and twice CH is equal to KH together with HS ; therefore KB together with BS will be to BS as KH together with HS is to HS ; and therefore KB will be to BS as KH to HS ; that is, as KB to SL ; therefore BS , SL , are equal; therefore the angle SBL is equal to the angle SLB ; that is, equal to the angle KBD ; and therefore, from a known property of the ellipse, BD is a tangent to the ellipse at the point B ; and therefore [32.] the sector ACB is greater than the sector ASE , and less than the sector ASF : but because G is the place of the planet, the area of the ellipse will be to the sector ASG as the periodic time of the

the planet round the sun is to the time elapsed since the planet passed through the point A; that is, as the area of the ellipse to the sector ACB; therefore the sector ASG is equal to the sector ACB; and therefore the sector ASG is greater than the sector ASE, and less than the sector ASF; therefore the line SG falls between the lines SE, SF; and therefore G, the place of the planet, is between the points E, F; therefore the planet has passed the point E, but is not come to the point F.

F I N I S.

the planet round the sun is, at the time of
 passed since the planet passed through the
 point A; that is, as the area of the ellipse
 to the sector ACB; therefore the sector
 ASO is equal to the sector ACB; and
 therefore the sector ASO is greater than the
 sector ASE, and less than the sector ASF;
 therefore the line SE is between the lines
 SE, SF; and the distance of the
 planet, is between the points E, F; there-
 fore the planet has passed the point E, but
 is not come to the point F.



C. I. N. 1. 2.

Fig. 115.

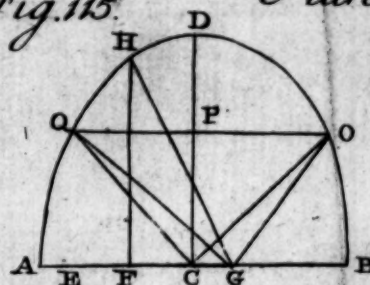


Plate 19.

Fig. 118.

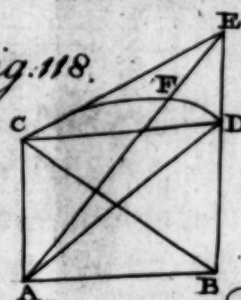


Fig. 119.

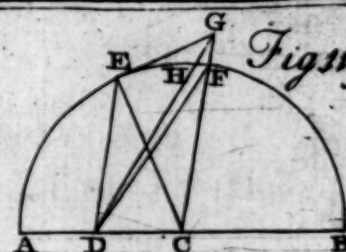


Fig. 116.

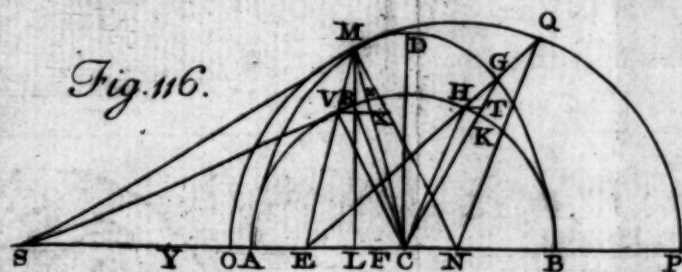


Fig. 120.

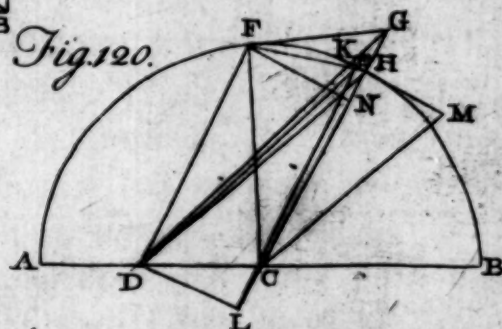


Fig. 117.

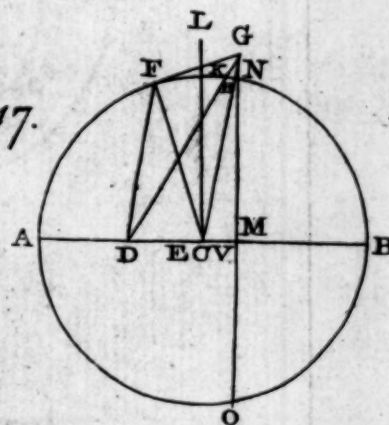


Fig. 121.

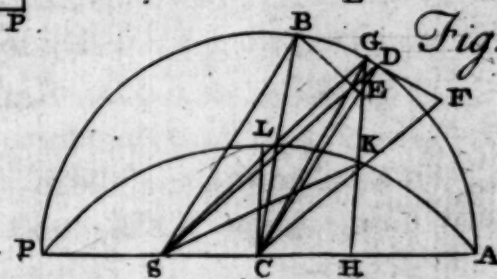


Fig. 122.

